Fall 2024 MATH33A Midterm 1 Review

Exercise 1. Re-write the following problems as augmented matrices representing linear systems:

(a) Solving the equation

$$-4x_1 + 3x_2 - 5x_4 = 3$$

$$x_1 - x_3 - x_4 = 7$$

$$-2x_1 + 3x_2 - 2x_3 - 7x_4 = 3$$

(b) Finding degree-2 polynomials $p(x) = ax^2 + bx + c$ such that $p(-2) = 9, p'(1) = 5$.

Exercise 2.

(a) List the rules for a matrix to be in reduced row echelon form.

(b) Use row reduction to solve the linear systems that you created augmented matrices for in Problem 1.

Exercise 3.

(a) Define what the rank of a matrix A is.

(b) What is the rank of $A = \begin{bmatrix} a & a^2 \\ a^3 & a^4 \end{bmatrix}$ in terms of a?

(c) What possible ranks can a 2×3 matrix A have? What form will the RREF's have for each possible rank?

Exercise 4.

(a) Define what a linear transformation is.

(b) If we know what a linear transformation does geometrically, how can we find the matrix of the linear transformation.

(c) Using this table if necessary for this part, find the matrix A representing reflection across the line $y = \frac{4}{3}x$ and the matrix B representing rotation by $\frac{4\pi}{3}$ (240 degrees). Describe A^{11}, B^{11} geometrically and what the entries of each are. Compute ABA algebraically and use your answer to describe what type of geometric transformation it represents.

Exercise 5.

(a) Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. What conditions are needed on m, n, p, q for the product AB to be defined? If the product is defined, what will be the dimensions of the matrix AB?

(b) If $T : \mathbb{R}^p \to \mathbb{R}^s$ is a linear transformation, how many rows and columns does the matrix A representing T have?

(c) Recall that square matrices A and B commute if $AB = BA$ (this is <u>not true in general</u> for square matrices). Turn "the set of matrices commuting with $\begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix}$" into a system of linear equations.

Exercise 6.

(a) What does it mean for an $m \times n$ matrix A to be invertible?

(b) How might you check that A is invertible and produce its inverse?

(c) If $n \times n$ matrices A, T satisfy $AT = I_n$, is A invertible? Is T invertible?

(d) For what values of k is $A = \begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ invertible? Give its inverse for those values. (Adapted from a textbook problem)

Exercise 7.

(a) Let A be an $m \times n$ matrix. What is the kernel of A? What is the image of A?

(b) Give a 3×2 matrix A whose image includes $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$.

(c) Find the kernel of $A = \begin{bmatrix} a & a^2 \\ a^3 & a^4 \end{bmatrix}$.

Exercise 8.

(a) Consider a set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$. What does it mean for $\vec{b}$ to be in the span of S (or a linear combination of vectors in S) and how can you check that? What does it mean for S to be linearly independent and how can you check that?

(b) What is a subspace of $\mathbb{R}^n$? Show that the set of vectors S in $\mathbb{R}^3$ whose coordinates add to 0 is a subspace and find a linearly independent set of vectors in S that spans S (i.e. a basis of S).

(c) Give a 3×2 matrix A whose image is the plane $5x + 3y - z = 0$. Show that the columns of A are linearly independent but the rows aren't.