Fall 2024 MATH 33A Midterm 2 Review

1 Computational Questions

Exercise 1. Find an orthonormal basis for $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} \right\}$ in $\mathbb{R}^4$. Extend your basis to an orthonormal basis for all of $\mathbb{R}^4$ by finding an orthonormal basis for $V^\perp$.

Solution First, perform the Gram-Schmidt process on the given vectors to get an orthonormal basis for V:

$$\boxed{\vec{u}_1 = \frac{1}{10} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - \text{proj}_{\vec{u}_1}\left(\begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \implies \boxed{\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} - \text{proj}_{\vec{u}_1}\left(\begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} \right) - \text{proj}_{\vec{u}_2}\left(\begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \implies \boxed{\vec{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}$$

Finally, $V^\perp$ is the vectors perpendicular to each of the vectors in either basis, i.e. solutions to

$$\left[\begin{array}{cccc|c} 1 & 7 & 1 & 7 & 0 \\ 0 & 7 & 2 & 7 & 0 \\ 1 & 8 & 1 & 6 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 7 & 0 \end{array}\right] \implies \boxed{\vec{u}_4 = \frac{1}{10} \begin{bmatrix} -7 \\ 1 \\ -7 \\ 1 \end{bmatrix}}$$

Exercise 2. Compute the QR factorization of $A = \begin{bmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{bmatrix}$.

Solution Gram-Schmidt gives the columns $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ of Q. Then R has (i, j) entry given by $\vec{u}_i \cdot \vec{v}_j$ where $\vec{v}_j$ is the jth column of A. So the factorization is

$$\begin{bmatrix} 1/10 & -1/\sqrt{2} & 0 \\ 7/10 & 0 & 1/\sqrt{2} \\ 1/10 & 1/\sqrt{2} & 0 \\ 7/10 & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Exercise 3. Compute the matrix of the orthogonal projection onto the solution space of

$$x_1 + x_2 - x_3 + x_4 = 0$$

in $\mathbb{R}^4$.

Solution This is equivalent to the RREF form

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \end{array}\right]$$

Therefore, the subspace has basis

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This basis can be made into an orthonormal basis by Gram-Schmidt:

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \vec{u}_3 = \frac{3}{2\sqrt{3}} \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{bmatrix}$$

The projection onto the subspace is given by

$$\text{proj}_V(\vec{v}) = (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\vec{u}_2 + (\vec{v} \cdot \vec{u}_3)\vec{u}_3$$

or equivalently it has matrix BB^T where B has columns $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Therefore, the projection is

$$\frac{1}{4} \begin{bmatrix} 3 & -1 & 1 & -1 \\ -1 & 3 & 1 & -1 \\ 1 & 1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{bmatrix}$$

Exercise 4. Find the least squares solution to $\begin{bmatrix} 3 & 2 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 4 & -6 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$. (This question is also on Worksheet 7)

Solution

$$A^T A = \begin{bmatrix} 10 & 10 & 0 & 0 \\ 10 & 21 & -22 & -11 \\ 0 & -22 & 44 & 22 \\ 0 & -11 & 22 & 11 \end{bmatrix}$$

and

$$A^T \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 24 \\ -22 \\ -11 \end{bmatrix}.$$

A solution to the system $A^T A\vec{x} = A^T \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is $\begin{bmatrix} 3/10 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. This is one, but not all least square solutions.

Exercise 5. Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Solution The determinant is

$$1 \cdot \det \begin{bmatrix} -2 & 4 & -8 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 2 & 4 & 8 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 2 & 4 & 8 \\ -2 & 4 & -8 \\ -1 & 1 & -1 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 2 & 4 & 8 \\ -2 & 4 & -8 \\ 1 & 1 & 1 \end{bmatrix}$$

If you similarly expand each of these determinants, the answer you'll get is

$$\boxed{\det(A) = 72}$$

Exercise 6. Consider the linear system given by

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

(a) Use Cramer's Rule to solve this system.

(b) Find the inverse of the above matrix in terms of its determinant and "classical adjoint" (adjugate). Use this inverse to solve the system.

(This question is copied from Worksheet 8)

Solution

(a) The determinant of the matrix is

$$2 \cdot \det \begin{bmatrix} 4 & 5 \\ 0 & 7 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 0 & 4 \\ 6 & 0 \end{bmatrix} = 146$$

The components of $\vec{x}$ are

$$x_1 = \frac{1}{146} \det \begin{bmatrix} 5 & 3 & 0 \\ 6 & 4 & 5 \\ 7 & 0 & 7 \end{bmatrix} = \frac{119}{146}, \qquad x_2 = \frac{1}{146} \det \begin{bmatrix} 2 & 5 & 0 \\ 0 & 6 & 5 \\ 6 & 7 & 7 \end{bmatrix} = \frac{164}{146}, \qquad x_3 = \frac{1}{146} \det \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 6 & 0 & 7 \end{bmatrix} = \frac{44}{146}$$

(b) The determinant of the matrix is

$$2 \cdot \det \begin{bmatrix} 4 & 5 \\ 0 & 7 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 0 & 4 \\ 6 & 0 \end{bmatrix} = 146$$

The classical adjoint is

$$\text{adj}(A) = \begin{bmatrix} 28 & -21 & 15 \\ 30 & 14 & -10 \\ -24 & 18 & 8 \end{bmatrix}$$

Therefore the inverse is

$$A^{-1} = \frac{1}{146} \begin{bmatrix} 28 & -21 & 15 \\ 30 & 14 & -10 \\ -24 & 18 & 8 \end{bmatrix}$$

and the solution to the system is

$$A^{-1}\vec{b} = \frac{1}{146} \begin{bmatrix} 28 & -21 & 15 \\ 30 & 14 & -10 \\ -24 & 18 & 8 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \frac{1}{146} \begin{bmatrix} 119 \\ 164 \\ 44 \end{bmatrix}$$

2 Conceptual Questions

Exercise 7.

(a) What is the transpose of $AB^{-2}C^T$?

(b) What is the definition of an orthogonal matrix in terms of the columns? What is the definition in terms of the transpose? Which of the following is enough for square matrix A to be orthogonal?

- $AA^T = I$
- $A^T A = I$
- A has orthonormal rows
- $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}, \vec{w}$

(c) Define what it means for a matrix to be symmetric in terms of the transpose. What conditions do you need on A for $AA^T - A^T A$ to be symmetric?

Solution

(a) $(AB^{-2}C^T)^T = (C^T)^T(B^{-2})^T A^T = C(B^T)^{-2}A^T$

(b) An orthogonal matrix is a square matrix with orthonormal columns. Equivalently, it is a square matrix A such that $A^T A = I = AA^T$. Actually, either of these equalities is sufficient (look back at Theorem 2.4.8 in the book!). The other two conditions in the bullet points are also equivalent to being orthogonal.

(c) A symmetric matrix is a matrix B satisfying $B^T = B$.

$$(AA^T - A^T A)^T = (AA^T)^T - (A^T A)^T = AA^T - A^T A$$

Therefore, $AA^T - A^T A$ is always symmetric!

Exercise 8.

(a) What is the relationship between the image of A and the matrix A^T? What is the relationship between the kernel of A and the matrix A^T?

(b) Is $\ker(A^TA) = \ker(A)$? Is Is $\ker(AA^T) = \ker(A)$?

(c) Use these relationships to show that $\mathrm{im}(A^TA) = \mathrm{im}(A^T)$.

(d) If the entries of two vectors are all strictly positive, what can you say about the angle between them?

Solution

(a) We have that $\mathrm{im}(A)^\perp = \ker(A^T)$. Replacing A with A^T we get $\mathrm{im}(A^T)^\perp = \ker(A)$.

(b) Yes, $\ker(A^TA) = \ker(A)$: if $A\vec{x} = 0$ then $A^TA\vec{x} = A^T\vec{0} = \vec{0}$; conversely if $A^TA\vec{x} = 0$ then $A\vec{x}$ is in the kernel of A^T (and so in $\mathrm{im}(A)^\perp$ by the last part) and in the image of A by the definition of the image of A, so $A\vec{x} = 0$ as desired ($\mathrm{im}(A) \cap \mathrm{im}(A)^\perp = 0$).

However, $\ker(AA^T) \neq \ker(A)$ in general. For instance, look at the example $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

(c) We have

$$\mathrm{im}(A^TA) = \ker((A^TA)^T)^\perp = \ker(A^TA)^\perp = \ker(A)^\perp = \mathrm{im}(A^T)$$

(d) If the entries of $\vec{v}, \vec{w}$ are all positive, then $\vec{v} \cdot \vec{w} > 0$. Remember that

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \cdot ||\vec{w}||}$$

where θ is the angle between $\vec{v}, \vec{w}$. Since cosine is only positive for angles $\theta < \frac{\pi}{2}$ (and we always choose θ to be at least 0 when we take the inverse cosine here), we know that $0 < \theta < \frac{\pi}{2}$, i.e. the angle between the vectors is acute.

Exercise 9. What is the determinant of the following matrices?

(a) The reduced row echelon form of a matrix A

(b) A noninvertible matrix

(c) $-A$ for A a 3×3 matrix

(d) $-A$ for A a 4×4 matrix

Solution

(a) The determinant will be 0 or 1 because the RREF will either be the identity matrix or have a row of all 0s. The determinant will also be related to the determinant of A as follows. Recall that the row operations to get to RREF are adding a multiple of a row to another row, multiplying a row by a scalar, and swapping rows. Multiplying a row by c changes the determinant of a matrix by c and swapping a row changes the determinant by a factor of (-1).

Adding a scalar multiple of a row to another does not change the determinant. Therefore, if RREF$(A) \neq I$ then $\det(A) = 0$ and if RREF$(A) = I$ then

$$\det(A) = (-1)^r \frac{1}{c_1} \cdots \frac{1}{c_k}$$

where r is the number of row swaps performed in the RREF and $c_1, \ldots, c_k$ are the scalars that you multiplied rows by during the RREF.

(b) 0

(c) Multiplying a matrix by -1 is the same as multiplying each row by (-1). Since a 3×3 matrix has 3 rows, the determinant is multiplied by a factor of $(-1)^3 = -1$. So the determinant is $-\det(A)$

(d) Multiplying a matrix by -1 is the same as multiplying each row by (-1). Since a 4×4 matrix has 4 rows, the determinant is multiplied by a factor of $(-1)^4 = 1$. So the determinant is $\det(A)$

Exercise 10.

(a) What's the determinant of an orthogonal matrix?

(b) Does there exist a 3×3 matrix A with $A^2 = -I_3$?

(c) If all the columns of a square matrix A are unit vectors, then is there a bound on the determinant of A?

(d) If $\det(A) = 1$, is A orthogonal?

(e) What is the relation between the 3×3 determinant and the cross product?

Solution

(a) If O orthogonal $O^T O = I$, so taking determinants of both sides we have

$$\det(O^T O) = \det(O^T)\det(O) = \det(O)^2 = 1$$

$$\implies \det(O) = \pm 1$$

(b) Notice that $-I$ is the same as I with the first row multiplied by -1, the second row multiplied by -1, and the third row multiplied by -1. Therefore, $\det(-I_3) = (-1)^3 = -1$. But $\det(A^2) = \det(A)^2$, so no such matrix exists because -1 does not have a real square root.

(c) Yes, the absolute value of the determinant must be less than or equal to 1. This is because the volume of a parallelepiped with side lengths 1 is at most 1.

(d) Not necessarily. $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$, but this matrix is not orthogonal.

(e) $\det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \vec{u} \cdot (\vec{v} \times \vec{w})$