# Fall 2024 MATH 33A Midterm 2 Review

## 1 Computational Questions

**Exercise 1.** Find an orthonormal basis for $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} \right\}$ in $\mathbb{R}^4$. Extend your basis to an orthonormal basis for all of $\mathbb{R}^4$ by finding an orthonormal basis for $V^\perp$.

**Exercise 2.** Compute the QR factorization of $A = \begin{bmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{bmatrix}$.

**Exercise 3.** Compute the matrix of the orthogonal projection onto the solution space of

$$x_1 + x_2 - x_3 + x_4 = 0$$

in $\mathbb{R}^4$.

**Exercise 4.** Find the least squares solution to $\begin{bmatrix} 3 & 2 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 4 & -6 & -3 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$. (This question is also on Worksheet 7)

**Exercise 5.** Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

**Exercise 6.** Consider the linear system given by

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

(a) Use Cramer's Rule to solve this system.

(b) Find the inverse of the above matrix in terms of its determinant and "classical adjoint" (adjugate). Use this inverse to solve the system.

(This question is copied from Worksheet 8)

## 2 Conceptual Questions

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**Exercise 7.**

(a) What is the transpose of $AB^{-2}C^T$?

(b) What is the definition of an orthogonal matrix in terms of the columns? What is the definition in terms of the transpose? Which of the following is enough for square matrix $A$ to be orthogonal?

- $AA^T = I$
- $A^T A = I$
- $A$ has orthonormal rows
- $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}, \vec{w}$

(c) Define what it means for a matrix to be symmetric in terms of the transpose. What conditions do you need on $A$ for $AA^T - A^T A$ to be symmetric?

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**Exercise 8.**

(a) What is the relationship between the image of $A$ and the matrix $A^T$? What is the relationship between the kernel of $A$ and the matrix $A^T$?

(b) Is $\ker(A^T A) = \ker(A)$? Is Is $\ker(AA^T) = \ker(A)$?

(c) Use these relationships to show that $\mathrm{im}(A^T A) = \mathrm{im}(A^T)$.

(d) If the entries of two vectors are all strictly positive, what can you say about the angle between them?

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**Exercise 9.** What is the determinant of the following matrices?

(a) The reduced row echelon form of a matrix

(b) A noninvertible matrix

(c) $-A$ for $A$ a $3 \times 3$ matrix

(d) $-A$ for $A$ a $4 \times 4$ matrix

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**Exercise 10.**

(a) What's the determinant of an orthogonal matrix?

(b) Does there exist a $3 \times 3$ matrix $A$ with $A^2 = -I_3$?

(c) If all the columns of a square matrix $A$ are unit vectors, then is there a bound on the determinant of $A$?

(d) If $\det(A) = 1$, is $A$ orthogonal?

(e) What is the relation between the $3 \times 3$ determinant and the cross product?

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