Fall 2024 MATH33A Worksheet 3: Sections 2.3, 2.4

Exercise 1. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) Compute A^{-1}.

(b) Use the inverse to find all solutions to $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, and all solutions to $A\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Exercise 2. Show that the following subsets are *not* subspaces of $\mathbb{R}^2$:

(a) $V = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$

(b) $V = \left\{ \begin{bmatrix} 3s+1 \\ 2-s \end{bmatrix} \mid s \in \mathbb{R} \right\}$

Show that the following subsets *are* subspaces of $\mathbb{R}^2$:

(c) $V = \left\{ \begin{bmatrix} t \\ 3s \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$.

(d) $V = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

Exercise 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of projection onto the line $y = x$. Is T invertible? Argue both (1) geometrically, and (2) algebraically by finding the matrix representation for T and computing its determinant.

Exercise 4. Let A be an $m \times n$ matrix, and let B be an **invertible** $n \times n$ matrix.

(a) Suppose that for all $\vec{b} \in \mathbb{R}^m$, $Ax = \vec{b}$ has a solution. What does this tell you about the image of A?

(b) How many solutions are there to $Bx = \vec{b}$ for any $\vec{b} \in \mathbb{R}^n$?

(c) What is the image of B (hint: does $Bx = \vec{b}$ always have a solution?)

(d) **(Challenge)** If B is invertible, what is $\mathrm{Im}(AB)$ in terms of the images of B, A? If C is an invertible $m \times m$ matrix, can you answer the same question for $\mathrm{Im}(CA)$?

Exercise 5. Find the inverse of the following matrix

$$A = \begin{bmatrix} 1 & c & c^3 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

in terms of $c \in \mathbb{R}$. Verify your answer with matrix multiplication.
