Unitary Representations of $SU(2)$ and $SL_2(\mathbb{R})$

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The topic of this paper is unitary representations of real Lie groups. When a group G is compact, we will see that in some sense we know everything we need to know about G from its ordinary (i.e. not a priori unitary) finite-dimensional representations. Moreover, unitary representations of compact groups will have some of the nice decompositions and character theory that we obtained from representations of finite groups. To illustrate the differences between unitary representation theory and the ordinary finite-dimensional representation theory, as well as the differences between the case of a compact group and a noncompact group, I will primarily consider two groups which have "the same" finite-dimensional complex representations: $SU(2)$ and $SL_2(\mathbb{R})$. (Note that I intend to produce a lot of the nice results for the compact case in detail, as we were only able to briefly touch on it at the end of Serganova's 261A class).

§1 Preliminaries

§1.1 Unitary Representations

Definition 1.1. For a Hilbert space H, the group $U(H)$ of **unitary operators** on H is comprised of bounded operators U on H such that $\langle Uv, Uw\rangle = \langle v, w\rangle$. A **(strongly continuous) unitary representation** of a Lie group G on Hilbert space H is a group homomorphism $\rho : G \to U(H)$ such that $g \mapsto \rho(g)\xi$ is continuous for all $\xi \in H$. For morphisms between unitary representations $f : (H_1, \rho_1) \to (H_2, \rho_2)$, we generally want to consider **bounded intertwining operators**: maps $f \in B(H_1, H_2)$ such that

$$\begin{array}{ccc} H_1 & \xrightarrow{f} & H_2 \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ H_1 & \xrightarrow{f} & H_2 \end{array}$$

Two representations are **unitarily equivalent** if there is a unitary (i.e. $\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1$) intertwining map between them. A representation is **irreducible** if its only closed G-invariant subspaces are 0 and V.

Note: By convention, Hilbert space inner products will be conjugate-linear in the first argument and linear in the second.

In the ordinary finite-dimensional case, if a Lie group G is not semisimple, in general we have no guarantee that its representations are, so classifying its indecomposable representations can be harder than knowing the irreducible ones. However, for finite-dimensional unitary representations we always have semisimplicity:

Theorem 1.2 (Complete Reducibility of Finite-Dimensional Unitary Representations)
For any Lie group G, any nonzero finite-dimensional unitary representation (H, ρ) of G decomposes as a direct sum of irreducible representations of G.

Proof. Suppose H_0 is a closed G-invariant subspace of H. Let $H_1 = H_0^\perp$, i.e.

$$H_1 = \{\zeta \in H : \langle \zeta, \xi \rangle = 0 \forall \xi \in H_0\}$$

Now, H_1 is G-invariant because $\langle g\zeta, \xi \rangle = \langle \zeta, g^{-1}\xi \rangle = \langle \zeta, \xi' \rangle = 0$. Furthermore, H_1 is closed as the intersection of closed sets

$$\bigcap_{\xi \in H_0} \{f_\xi^{-1}(0) : f_\xi(\zeta) = \langle \zeta, \xi \rangle\}$$

Thus, $H = H_0 \oplus H_1$, a direct sum of closed subrepresentations. For all finite-dimensional H, this gives that H is a direct sum of irreducibles by induction. □

In the greatest generality, the same is not necessarily true in infinite dimensions, and so the best that is possible is a "direct integral" decomposition (see e.g. [Deitmar] for a more detailed discussion of singular integrals). However, we can decompose the Hilbert space of a unitary representation into subspaces with "cyclic vectors" — a subspace V has a cyclic vector ξ if the closure of the span of $\{\pi(g)\xi : g \in G\}$ is V — as is often the case in functional analysis for single operators. Assuming the axiom of choice, this can be shown via Zorn's Lemma: the collection of pairwise orthogonal cyclic subspaces satisfies the chain condition (the union of an ascending chain will still consist of pairwise orthogonal cyclic subspaces) and so there is some maximal collection C. Then,

$$H = \bigoplus_{(\xi, V_\xi) \in C} V_\xi$$

because as in the above theorem we can find an element ξ' of the orthogonal complement of this direct sum, whose cyclically-generated subspace must then be orthogonal (because the orthogonal complement is closed) to our previous subspaces, giving a contradiction.

§1.2 Background Results

Before proceeding, I'll collect here some background results which I will need later. The omitted proofs for results in functional analysis and integration theory can be found in [Lang: Real]. For the proof of Schur's Lemma I will follow the outline given in [Folland].

Theorem 1.3 (Spectral Theorem for Compact Self-Adjoint Operators)
If T is a compact self-adjoint operator on Hilbert space H, then there exists an orthonormal eigenbasis $\{e_i\}$ of H with real eigenvalues $\lambda_i \in \mathbb{R}$.

Theorem 1.4 (Haar Measure)
There exists a left-invariant Borel measure μ on any locally-compact group G. If G is compact or semisimple then μ is also left-invariant.

Lemma 1.5 (Schur's Lemma)

Let $(H_1, \rho_1), (H_2, \rho_2)$ be two irreducible unitary representations of G and T a bounded intertwining operator. Then $T = 0$ or a scalar multiple of one fixed intertwiner (the identity for $H_1 = H_2$).

Proof. Suppose $T : H_1 \to H_2$ is a nonzero bounded intertwining operator. Then, T^* is bounded and so we have

$$T^*\rho_2(g) = T^*\rho_2(g^{-1})^* = (\rho_2(g^{-1})T)^* = (T\rho_1(g^{-1}))^* = \rho_1(g)T$$

using the fact that

$$\langle \rho_i(g)x, y\rangle_i = \langle \rho_i(g^{-1})\rho_i(g)x, \rho_i(g^{-1})y\rangle_i = \langle x, \rho_i(g^{-1})y\rangle_i \implies \rho_i(g)^* = \rho_i(g^{-1})$$

This means that T^* is a bounded intertwining operator, and so T^*T is a bounded intertwining operator on H_1, and similarly for TT^* on H_2. So, let S be a bounded intertwiner on H_i. By the same argument as for T, S^* is a bounded intertwiner, and so $\text{Re}(S) = \frac{1}{2}(S + S^*)$ and $\text{Im}(S) = \frac{1}{2i}(S - S^*)$ are self-adjoint intertwiners. Note that if both were multiples of the identity we'd have

$$\text{Re}(S) = \lambda_1 I \quad \text{Im}(S) = \lambda_2 I \implies S = \lambda_1 I + i\lambda_2 I = \lambda_3 I$$

And in fact both are multiples of the identity for operator theoretic reasons: if a set of operators (here, $\{\rho(g)\}$) commutes with a self-adjoint operator (here, the real and imaginary parts of S), then it commutes with the functional calculus of any bounded function on the spectrum of the operator. So, if one of the operators, say $\text{Re}(S)$ were not a multiple of the identity, it has some nontrivial spectral projection which commutes with G. Finally, this would contradict irreducibility because for any projection P commuting with G, its range (which is not all of H_i) is closed and G-invariant:

$$y = Px \implies gy = gPx = P(gx)$$

Now, using that nonzero bounded intertwiners on a single space are multiples of the identity, that means $TT^* = T^*T = \lambda I$, i.e. $\frac{1}{\sqrt{\lambda}}T$ is unitary, so T is a unitary equivalence. For two unitary equivalences, T_1, T_2, $T_1^*T_2$ is a bounded intertwiner on H_1 and so $T_1^*T_2 = \lambda' I$, giving $T_2 = \lambda' T_1$. □

§2 $SU(2)$ and $SL_2(\mathbb{R})$

Our main objects of study are the following real Lie groups:

$$\boxed{SL_2(\mathbb{R}) := \{A \in GL_2(\mathbb{R}) : \det A = 1\}}$$

$$\boxed{SU(2) := \{U \in Mat_{2\times 2}(\mathbb{C}) : U^*U = 1, \det U = 1\}}$$

We will shortly see that $SU(2)$ and $SL_2(\mathbb{R})$ have the same finite-dimensional representations. Therefore, one might suspect that the same will be true of their unitary representations. However, this is not the case and the remaining chapters will be devoted to highlighting the differences in their unitary representation theories. The key difference is the following

Theorem 2.1
$SU(2)$ is compact, whereas $SL_2(\mathbb{R})$ is not.

Proof. We can see that $SL_2(\mathbb{R})$ is not compact because

$$\det\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix} = 1, \quad \left|\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\right|_{\mathbb{R}^4} = \sqrt{t^2+2}$$

which can be arbitrarily large. However,

$$A \in SU(2) \iff ad-bc = |a|^2+|b|^2 = |c|^2+|d|^2 = |a|^2+|c|^2 = |b|^2+|d|^2 = 1, a\bar{c}+b\bar{d} = \bar{a}b+\bar{c}d = 0$$

$$\iff A = \begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix} \text{ with } |a|^2+|b|^2 = 1$$

which is topologically just a 3-sphere. □

§2.1 Finite-Dimensional Representations of $SU(2)$ and $SL_2(\mathbb{R})$

To contextualize the choice of the groups $SU(2)$ and $SL_2(\mathbb{R})$, I'll give a overview of why they have the same finite-dimensional complex representation theory. Since this is not my main focus, I will attempt to remain brief; further details can be found in [Knapp].

An important corollary of the $SU(2)$ being topologically a 3-sphere is that it is a simply-connected Lie group, and so from the general theory we know that its finite-dimensional representation theory is equivalent to the representation theory of its Lie algebra $\mathfrak{su}(2)$. Since $SU(2)$ is a matrix group, its Lie algebra is straightforward to calculate:

$$\mathfrak{su}(2) = \{x \in \mathfrak{gl}_2(\mathbb{C}) : \exp(\mathbb{R}x) \in SU(2)\} = \{x : \exp(\mathbb{R}x)\exp(\mathbb{R}^x)^* = 1, \det(\exp(\mathbb{R}x)) = 1\}$$

Now, $\det(\exp(A)) = e^{\mathrm{Tr}(A)}$ and using the Baker-Campbell-Hausdorff Formula we can expand $\exp(tx)\exp(tx)^*$ as

$$\exp(t(x+x^*) + \frac{1}{2}[tx, tx^*] + \dots) = 1$$

Differentiating with respect to t and evaluating at $t = 0$, we're just left with

$$\mathfrak{su}(2) = \{x \in \mathfrak{gl}_2(\mathbb{C}) : \mathrm{Tr}(x) = 0, x + x^* = 0\}$$

The fact that $SU(2)$ and $SL_2(\mathbb{R})$ have the same finite-dimensional complex representation theory is closely related to the fact that their Lie algebras both complexify to $\mathfrak{sl}_2(\mathbb{C})$. The simplest way to see this is from involutions on $\mathfrak{sl}_2(\mathbb{C})$: from any antilinear bracket-preserving involution we can recover a decomposition $\mathfrak{g} = k \oplus ik$ and vice versa via the fixed points of the involution. In our case, the involutions on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ are

$$\sigma_1\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{pmatrix} \qquad \sigma_2\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}-\bar{a} & -\bar{c}\\ -\bar{b} & -\bar{d}\end{pmatrix}$$

corresponding to $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}(2)$ respectively. Note that in general every complex semisimple Lie algebra has exactly one compact form (a real Lie algebra with negative-definite Killing form that complexifies to the given semisimple algebra), via an involution acting exactly as σ_2 but on $\mathfrak{sl}_2$-triples in the algebra. Now, from these decompositions, we can isomorphically map

$$\mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{sl}_2(\mathbb{C}) \qquad \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{sl}_2(\mathbb{C})$$

via $x \otimes \lambda \mapsto \lambda x$ acting on the real and complex parts as in normal multiplication of complex numbers. Since representations of Lie algebras are linear objects (as opposed to representations of Lie groups), the complex representations of a real Lie algebra are exactly the complex representations of its complexification (for instance, we can extend by the change of scalars formulation and recover the representation of the real algebra from the involution). Note that complete reducibility of $\mathfrak{sl}_2(\mathbb{R})$ can be shown in exactly the same way as we have shown for $\mathfrak{sl}_2(\mathbb{C})$ with the construction of a Casimir element.

One problem remains: the representation theory of $SL_2(\mathbb{R})$ is not equivalent to that of $\mathfrak{sl}_2(\mathbb{R})$ because $SL_2(\mathbb{R})$ is not simply-connected! One way of explicitly seeing this is the following. Any semisimple real Lie group will have an "Iwasawa decomposition" as products from compact, abelian, and nilpotent subgroups, in our case

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

which is exactly the "QR decomposition" from linear algebra. The map $SL_2(\mathbb{R}) \to K \times A \times N$ is not a group homomorphism, but it is a homeomorphism, and so we can extract the topological structure of $SL_2(\mathbb{R})$: $K = S^1, A = \mathbb{R}^{>0} \cong \mathbb{R}, N = \mathbb{R}$. Therefore, the fundamental group of $SL_2(\mathbb{R})$ is $\mathbb{Z}$. In general, the fundamental group lies will be the center of the universal covering group, and so there is a simply-connected group $\widetilde{SL_2(\mathbb{R})}$ with center $\mathbb{Z}$ which covers $SL_2(\mathbb{R})$. However, this is not an algebraic group (and $SL_2(\mathbb{R})$ has no algebraic central extensions) because it has no faithful finite-dimensional representations, which can be shown to imply that the finite-dimensional irreducible representations of $SL_2(\mathbb{R})$ are the same as $SL_2(\mathbb{C})$ (see [Bourbaki]).

Explicitly, we can construct the finite-dimensional irreducible representations in a similar way as we can for $SL_2(\mathbb{C})$: let $V_n \subset \mathbb{C}[x, y]$ be the $(n+1)$-dimensional space of homogeneous polynomials of degree n. Then, we can construct both groups' irreducible actions on V_n by the same formula:

$$\rho(\begin{pmatrix} a & b \\ c & d \end{pmatrix})(P(x, y)) = P(A^{-1}(x, y)) = P(dx - by, -cx + ay)$$

§3 The Compact Case: Unitary Representations of $SU(2)$

Now, we can examine the unitary representation theory of $G = SU(2)$. For all of the results of this section, the chiefly important detail is that G is compact, so we will work in that generality. Similar arguments to the ones I will carry out are done in [Folland] and [Gruson].

The first significant fact is that we won't be able to build any new representations than ones we could from considering finite-dimensional representations:

Theorem 3.1

All irreducible unitary representations of a compact group G are finite-dimensional, and any unitary representation is a direct sum of irreducibles.

Proof. Suppose (V, ρ) is an irreducible unitary representation of G. Choose an arbitrary $v \in V$ with $||v|| = 1$ and let T be the projection onto v:

$$T(x) := \langle v, x \rangle v$$

Note that this T is self-adjoint and positive because it's an orthogonal projection. Define operator

$$Q(x) = \int_G (gTg^{-1})(x)\, dg$$

Now, Q is self-adjoint because

$$\langle Q(x), y\rangle = \langle \int_G (gTg^{-1})(x)\, dg, y\rangle = \int_G \langle (gTg^{-1})(x), y\rangle\, dg = \int_G \langle g^{-1}(gTg^{-1})(x), (g^{-1})(y)\rangle\, dg$$

$$= \int_G \langle (g^{-1})x, (T^* g^{-1})y\rangle\, dg = \int_G \langle x, (gTg^{-1})y\rangle\, dg = \langle x, Q(y)\rangle$$

Q is indeed bounded because unitary operators preserve norm and projections are norm-decreasing, giving

$$||Q(x)|| \leq \int_G ||(gTg^{-1})(x)||\, dg = \int_G ||T(x)||\, dg < \int_G ||x||\, dg = ||x||$$

and furthermore, we will show Q is compact. The function $f : G \to H$ by $f(g) = gv$ is uniformly continuous by compactness of G. Therefore, for an $\epsilon > 0$ we can find a $\delta > 0$ and finitely-many δ-balls $B(g_1, \delta), \ldots, B(g_m, \delta)$ such that for $g \in B_i$, $||gv - g_i v|| < \frac{\epsilon}{2}$. Make these balls into disjoint B_i (by arbitrarily taking complements). Then, for $g \in B_i$,

$$||(gTg^{-1})(x) - (g_i T g_i^{-1})(x)|| = ||\langle gv, x\rangle gv - \langle g_i v, x\rangle g_i v|| =$$

$$= ||\langle gv - g_i v, x\rangle gv + \langle g_i v, x\rangle (g - g_i)v|| \leq ||\langle gv - g_i v, x\rangle gv|| + ||\langle g_i v, x\rangle (g - g_i)v||$$

$$< \frac{\epsilon||x||}{2} + \frac{\epsilon||x||}{2} = \epsilon||x||$$

using that $||v|| = 1$. That means, we get a sequence of maps converging to Q in norm by defining

$$Q_\epsilon(x) = \sum_i \int_{B_i} (g_i T g_i^{-1})(x)\, dg = \sum_i (g_i T g_i^{-1})(x) \cdot dg(B_i)$$

which is finite-rank because the image is spanned by $\{g_1 v, \ldots, g_m v\}$. This shows compactness.

By the Spectral Theorem for compact self-adjoint operators, Q has a basis of eigenvectors $\{e_i\}$ with real eigenvalues λ_i. Q is positive because T is:

$$\langle Q(x), x\rangle = \int_G \langle (gTg^{-1})(x), x\rangle\, dg = \int_G \langle T(g^{-1}x), g^{-1}x\rangle\, dg \geq 0$$

This means that the eigenvalues of Q are nonnegative:

$$\lambda = \lambda\langle e_i, e_i\rangle = \langle Qe_i, e_i\rangle \geq 0$$

Now, if Q has only eigenvalues 0, it must be the 0 operator because we have a basis of eigenvectors. However, Q is not zero because

$$\langle Q(v), v\rangle = \int_G \langle (gTg^{-1})(v), v\rangle\, dg = \int_G \langle g\langle v, g^{-1}v\rangle v, v\rangle\, dg = \int_G \langle gv, v\rangle^2\, dg > 0$$

So, let λ be a positive eigenvalue of Q. Define $W = \text{Ker}(Q - \lambda I)$. We can see that W is G-invariant because again using invariance of the Haar measure we have

$$(Q - \lambda I)(g_0 w) = \int_G (gTg^{-1})(g_0 w)\, dg - g_0 \lambda w = \int_G (gT(g_0 g)^{-1})(w)\, dg - g_0 \lambda w$$

$$= \int_G (g_0 g T g^{-1})(w)\, dg - g_0 \lambda w = g_0(Q - \lambda I)(w) = 0$$

Since W is also closed (it's a kernel), irreducibility gives that $W = V$. For any orthonormal set $\{e_1, \ldots, e_n\}$ in W, we have that

$$\sum_{i=1}^{n} \langle e_i, Qe_i \rangle = \sum_{i=1}^{n} \int_G \langle g^{-1} e_i, (Tg^{-1})(e_i) \rangle \, dg$$

The action of G takes orthonormal sets to orthonormal sets and for any orthonormal set $\{f_i\}$ we have

$$\sum_{i=1}^{n} \langle f_i, Tf_i \rangle = \sum_{i=1}^{n} \langle f_i, \langle v, f_i \rangle v \rangle = \sum_{i=1}^{n} \langle f_i, v \rangle^2 \leq 1$$

Therefore, since the Haar measure was normalized to have measure 1 on the group we get

$$\sum_{i=1}^{n} \langle e_i, Qe_i \rangle \leq 1$$

By definition of W as the given kernel, we can complete the proof with the estimate

$$\sum_{i=1}^{n} \langle e_i, \lambda e_i \rangle = \lambda \sum_{i=1}^{n} \langle e_i, e_i \rangle = \lambda n \leq 1 \implies n \leq \frac{1}{\lambda}$$

and in particular, this gives an upper bound on the dimension of $W = V$.

Now, when (V, ρ) is an arbitrary unitary representation, the construction of Q and W remains the same as before, and we still have a finite upper bound on the dimension of W, so W is a finite-dimensional subrepresentation which we can decompose into irreducibles. We can then use Zorn's Lemma as before to get that V must be a direct sum of irreducibles: for any maximal collection of pairwise orthogonal irreducible invariant subspaces, if its direct sum is not all of V then we could find a finite-dimensional subrepresentation to decompose into irreducibles in the orthogonal complement, contradicting maximality. □

From this theorem, we know that unitary representations of $SU(2)$ are built out of the same atoms as in the finite-dimensional case. To complete the classification of these atoms, two questions still remain: can all finite-dimensional representations be made into unitary representations and is there more than one way to do so (up to unitary equivalence)? The answers are yes and no, respectively:

Theorem 3.2

If G is compact, every finite dimensional representation (V, ρ) possesses a Hermitian inner product $\langle \cdot, \cdot \rangle$ such that $(V, \langle \cdot, \cdot \rangle, \rho)$ is unitary. If (V, ρ) is an irreducible representation, then up to re-scaling there is a unique Hermitian inner product for which the representation is unitary.

Proof. Let (V, ρ) be a finite-dimensional representation of compact Lie group G and equip V with arbitrary inner product $\langle \cdot, \cdot \rangle_0$. Define

$$\langle v, w \rangle = \int_G \langle gv, gw \rangle_0 \, dg$$

using the right Haar measure dg. Then, conjugate-linearity of the inner product, linearity of the action of g, and linearity of the integral give that $\langle \cdot, \cdot \rangle$ is conjugate-linear. For $v \neq 0$,

$$\langle v, v \rangle = \int_G \langle gv, gv \rangle_0 \, dg > 0$$

since it is an integral of positive terms (g acts by invertible maps, so $g^{-1}v \neq 0$). Thus $\langle \cdot, \cdot \rangle$ defines an inner product on V. Finally, for any $h \in G$,

$$\langle hv, hw \rangle = \int_G \langle ghv, ghw \rangle_0 \, dg = \int_G \langle gv, gw \rangle_0 \, dg = \langle v, w \rangle$$

by right-invariance of the Haar measure. In particular, (V, ρ) is a unitary representation on $(H, \langle \cdot, \cdot \rangle)$.

Now, suppose V is irreducible and $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ are two G-invariant inner products on V, i.e. two inner products for which G is unitary. Pick an orthonormal basis of V with respect to $\langle \cdot, \cdot \rangle_1$ using the Gram-Schmidt process. Let T be the matrix of the inner product $\langle \cdot, \cdot \rangle_2$ with respect to our orthonormal basis (i.e. with matrix coefficients $T_{ij} = \langle e_j, e_k \rangle_2$). Then, since $\langle \cdot, \cdot \rangle_2$ is G-invariant, we know

$$\rho(g)^* T \rho(g) = T$$

where the adjoint is taken with respect to our basis. Then, $\rho(g)^*$ is a unitary matrix in our orthonormal basis because of G-invariance of $\langle \cdot, \cdot \rangle_1$. Thus,

$$T\rho(g) = \rho(g)T$$

We assumed that V is irreducible, so by Schur's Lemma we have that $T = \lambda I$ as desired. □

From this theorem, we know the problem of finding irreducible unitary representations of a compact group G is equivalent to finding its irreducible complex finite-dimensional representations, which we have already done! Even more, we will see that much like the representations of finite groups, we can find all irreducible representations inside a regular representation.

Definition 3.3. For a Lie group G, let $H = L^2(G, dg)$ equipped with the standard inner product

$$\langle f_1, f_2 \rangle = \int_G \bar{f}_1(g) f_2(g) \, dg$$

The **left-regular representation of G** is the unitary representation

$$\left(L_g(f)\right)(h) = f(g^{-1}h)$$

and similarly the **right-regular representation** R_g has value $f(gh)$. Each is well-defined (for the proper Haar measure) because the squared-integral is invariant under shifts by elements of the group. Moreover, we can see that each is unitary because again by invariance of the Haar measure,

$$\int_G \bar{f}_1(hg^{-1}) f_2(hg^{-1}) \, dh = \int_G \bar{f}_1(h) f_2(h) \, dh$$

Now, suppose (V, ρ) is any unitary representation of a compact group G. For any $w \in V$, we can form the **matrix coefficient** map given by

$$F_v(w) = F_{v,w} = \Big(g \mapsto \langle v, gw\rangle\Big)$$

whose image is in $L^2(G)$ because the action of g and $\langle v, \cdot\rangle$ are continuous and continuous functions on a compact set are square-integrable. Then, F is a map of representations because

$$R_g F_{v,w}(h) = F_{v,w}(hg) = \langle v, hgw\rangle = F_{v,gw}(h)$$

and so F_v is an intertwining operator with respect to the given action on V and the right-regular representation. Additionally, F_v is nonzero for nonzero v because

$$F_v(v) = \Big(g \mapsto \langle v, gv\rangle\Big) \implies F_{v,v}(e) = ||v||^2$$

If V is irreducible, Schur's Lemma tells us that F_v is an isomorphism onto its image for any nonzero v, and so every irreducible representation of G can be found inside $L^2(G)$. In fact:

Theorem 3.4 (Peter-Weyl Theorem)

Matrix coefficients of irreducible representations of G are orthogonnal and span a dense subspace of $L^2(G)$. In particular, we have the following Hilbert space direct sum (i.e. the closure of the algebraic direct sum) decomposition:

$$L^2(G) = \bigoplus_{V \text{ irred}} \mathbb{C}^{\dim V} \otimes V$$

Proof. First, for a single irreducible unitary representation (V, ρ), suppose we take two different choices of vectors $v, w, v', w' \in V$. Defining similar operators as before:

$$T(x) = \langle v, x\rangle v' \qquad Q(x) = \int_G (gTg^{-1})(x)\, dg$$

is an intertwining map on V by the same argument as before. By Schur's Lemma, $Q = \lambda I$. Since V is finite-dimensional, we can take the trace of Q and T, giving

$$\lambda \dim V = \operatorname{Tr} Q = \sum_i \int_G \langle e_i, (gTg^{-1})(e_i)\rangle\, dg = \sum_i \int_G \langle g^{-1}e_i, (Tg^{-1})(e_i)\rangle\, dg = \operatorname{Tr} T$$

since the image of an orthonormal basis under g^{-1} is orthonormal. If we include v in some orthonormal basis, we can see

$$\lambda \dim V = \sum_i \langle e_i, \langle v, e_i\rangle v'\rangle = \langle v, v'\rangle \implies \lambda = \frac{\langle v, v'\rangle}{\dim V}$$

Finally,

$$\frac{1}{\dim V}\langle v, v'\rangle\langle w', w\rangle = \langle w', Qw\rangle = \int_G \langle w', \langle v, g^{-1}w\rangle gv'\rangle\, dG = \int_G \langle v, g^{-1}w\rangle\langle w', gv'\rangle\, dG$$

$$= \int_G F_{v,w}(g^{-1})F_{v',w'}(g)\, dg = \int_G \overline{F_{v,w}(g)}F_{v',w'}(g)\, dg = \langle F_{v,w}, F_{v',w'}\rangle_{L^2}$$

We can use the same method for nonisomorphic irreducible representations V, V': T is now an operator from V to V' and Q is an intertwining map and so by Schur's Lemma $Q = 0$; using the last line of calculation in reverse with $\lambda = 0$, the matrix coefficients are orthogonal. Note that these orthogonality relations almost immediately give that the characters of G are orthonormal.

Now, let (V, ρ) be an irreducible unitary representation. Define $\Phi : V \to \mathrm{Hom}_G(V, L^2(G))$ via

$$\Phi(v) = \Big(w \mapsto F_{v,w}\Big)$$

This Φ is anti-linear because

$$F_{\lambda_1 v + \lambda_2 v', w}(g) = \overline{\lambda_1}\langle v, gw\rangle + \overline{\lambda_2}\langle v', gw\rangle$$

For $v \neq 0$, $F(v) \neq 0$ because as previously shown $F_{v,v}(e) = ||v||^2$, and so Φ is injective. Moreover, $f \in \mathrm{Hom}_G(V, L^2(G))$ means that f is linear and $f(v)(gh) = f(hv)(g)$. The map $w \mapsto f(w)(e)$ is therefore a continuous linear functional on V, and so by the Riesz Representation Theorem there exists $v \in V$ such that $f(w)(e) = \langle v, w\rangle$. Putting these two properties together, $f(w)(g) = f(gw)(e) = \langle v, gw\rangle = F_{v,w}(g)$. Thus, Φ is bijective, and so (e.g. by reversing the role of v and w) we can obtain a vector space isomorphism between V and $\mathrm{Hom}_G(V, L^2(G))$, so $\mathrm{Hom}_G(V, L^2(G))$ is the desired multiplicity space. Forming

$$H = \overline{\bigoplus_{V \text{ irred}} \mathrm{Hom}_G(V, L^2(G)) \otimes V}$$

we can see that $H = L^2(G)$ because otherwise its orthogonal complement would be a subrepresentation of $L^2(G)$ and contain some irreducible finite-dimensional subrepresentation, which is in the algebraic direct sum, giving a contradiction (by matrix coefficient orthogonality). □

§4 Unitary Representations of $SL_2(\mathbb{R})$

With a very nice theory developed for the compact $SU(2)$, the goal in this final chapter will be to demonstrate ways the analogous statements fail when working with the noncompact $SL_2(\mathbb{R})$. The main sources for this section will be [Knapp] and [Lang: $SL_2(\mathbb{R})$].

Theorem 4.1

The only finite-dimensional unitary representation of $SL_2(\mathbb{R})$ are the trivial ones.

Proof. Let (H, ρ) be a finite-dimensional unitary representation of $SL_2(\mathbb{R})$. Its kernel $G' \subset G$ must be a closed normal subgroup of G because ρ is a continuous group homomorphism.

Define

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_k = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix} \in SL_2(\mathbb{R})$$

Then, for $k \neq 0$ we have

$$B_k^{-1} A^k B_k = \begin{pmatrix} \frac{1}{\sqrt{k}} & 0 \\ 0 & \sqrt{k} \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A$$

Since $\rho(A)$ is unitary, the finite-dimensional spectral theorem gives that $\rho(A)$ is diagonalizable with (possibly repeated) eigenvalues $\lambda_1, \dots, \lambda_n$. Therefore,

$$\operatorname{Tr} \rho(A^k) = \operatorname{Tr} \rho(B_k A B_k^{-1}) = \operatorname{Tr}(\rho(B_k)\rho(A)\rho(B_k)^{-1}) = \operatorname{Tr} \rho(A)$$

$$\implies \lambda_1 + \cdots + \lambda_n = \lambda_1^k + \cdots + \lambda_n^k$$

for all $k \neq 0$ and in particular for $k > n$. This means we must have $\lambda_1 = \cdots = \lambda_n = 1$, and so $A \in G'$. However, I claim the only nontrivial proper normal subgroup of $SL_2(\mathbb{R})$ is $\{\pm I\}$. This lemma implies $G' = G$ and so the representation is trivial. □

Lemma 4.2

The only nontrivial proper normal subgroup of $SL_2(\mathbb{R})$ is $\{\pm I\}$.

Proof. Consider the standard action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$ by matrix-vector multiplication. Because this action is linear it descends to an action on $\mathbb{RP}^1$, the set of lines in $\mathbb{R}^2$. For any four of these lines spanned by, say, v_1, v_2, v_3, v_4, then possibly changing signs of one of the v_i, there is a matrix A such that $Av_1 = v_3, Av_2 = v_4$ with positive determinant, and so $A' = \frac{1}{\sqrt{\det A}} A \in SL_2(\mathbb{R})$ with $A'(\mathbb{R}v_1) = v_3, A'(v_2) = v_4$, a stronger form of transitivity.

Fix $p = [1:0]$, which by transitivity satisfies $\mathbb{R}^2 = G/\mathrm{Stab}(p)$ as sets. First, we will show this $G_p = \mathrm{Stab}(p)$ is maximal in the sense that it is only properly contained in G. Suppose for the sake of contradiction there were some strict inclusion $G_p \subset G_0 \subset G$, then for $g \notin G_p, g \in G_0$ and $g_0 \notin G_0$ by our transitivity calculation on $\mathbb{R}^2 = G/G_p$ there must exist $h \in G$ such that $hG_p = G_p$ and $h(gG_p) = g_0 G_p$. This means h stabilizes p and so $hg \in G_0$. This is a contradiction because our last coset equation then gives $g_0^{-1} hg \in G_p \subset G_0$, meaning $g_0 \in G_0$. So, we have shown G_p is maximal.

Now, recall $N := \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\} \subset G_p$ from our Iwasawa decomposition and suppose G' is any normal subgroup of G. We will show now that G' is a subgroup of G_p. Suppose for the sake of contradiction G' is not a subgroup of G_p. Then we must have $G_p \subsetneq G_p G'$ and so by maximality $G_p G' = G$. In the quotient $\pi : G \to G/G'$, this means $\pi(G_p) = \pi(G)$ and so $\pi(NG') = \pi(N)$, meaning $\pi(NG')$ is normal in $\pi(G)$ by normality of N in G. In particular, conjugating N by the matrix swapping the two coordinates, we know $N' = \{\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R}\} \subset NG'$. Then, N and N' generate G by the LU decomposition from linear algebra. Thus, $NG' = G$. Now, N is abelian (the group operation is addition in the upper right entry), meaning $G/G' = N/(N \cap G')$ is abelian, i.e. $[G,G] \subset G'$. Furthermore, $N, N' \subset [G,G]$ via

$$\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & r \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & x + r^2 x \\ 0 & 1 \end{pmatrix}$$

since $x + r^2 x = t$ has solutions for any t. Thus, $[G,G] = G$ and so $G' = G$. This is a contradiction, so G' is a subgroup of G_p.

Finally, this tells us that G' stabilizes p and so by normality $G' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ stabilizes $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p = [0:1]$. The only elements which can stabilize both $[0:1]$ and $[1:0]$ are $\pm I$, so we are done. □

As a corollary to this theorem, we know that there is no overlap between irreducible unitary representations of $SL_2(\mathbb{R})$ and of $SU(2)$. There is still a canonical representation of $SL_2(\mathbb{R})$, namely the regular representation of $L^2(G)$, and one might expect that we can still find all irreducible representations of G inside of $L^2(G)$. We will see that this is not true: there are two classes of irreducible representations which are in $L^2(G)$ and one which is not.

§4.1 Action by Möbius Transformations

To explicitly construct the irreducible representations of G, it will be necessary to consider another natural action of G, this time on the upper half plane $\mathcal{H}$ of the complex numbers. In particular, let $SL_2(\mathbb{R})$ act by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

This map is well-defined because $cz+d=0$ would require $z=\frac{-d}{c}$ to be real, and does in fact map $\mathcal{H}$ to itself because for $z=z_1+iz_2$,

$$\operatorname{Im}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az_2cz_1+az_2d-az_1cz_2-cz_2b}{(cz_1+d)^2+c^2z_2^2} = \frac{z_2}{(cz_1+d)^2+c^2z_2^2} > 0$$

This action is transitive:

$$\frac{\sqrt{z_2}i+\frac{z_1}{\sqrt{z_2}}}{\frac{1}{\sqrt{z_2}}} = z_2i+z_1 = z; \text{ and } ad-bc = \sqrt{z_2}\frac{1}{\sqrt{z_2}} - 0 = 1$$

The stabilizer of the imaginary unit can be computed as follows:

$$i = \frac{ai+b}{ci+d} \implies di-c = ai+b \implies d=a, b=-c$$

Conversely, we see

$$\frac{ai+b}{-bi+a} = (ai+b)\frac{a+bi}{a^2+b^2} = \frac{ia^2+ib^2}{a^2+b^2} = i$$

Therefore, the stabilizer is $\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2+b^2=1\}$, i.e. the set of rotation matrices, R_θ which are topologically form a circle. In particular, $SL_2(\mathbb{R})/S^1$ acts freely on the upper half-plane.

Now, it is a standard result from Riemannian geometry (see [Lee]) that any action of a Lie group that is proper — meaning the pre-images under $(g,x)\mapsto(g\cdot x,x)$ of compact sets in $X\times X$ are compact in $G\times X$ — has an invariant metric. Properness is satisfied here because if a sequence escapes to infinity in $G\times X$ (in the Euclidean metric on X), then its image must as well (e.g. by the preceding calculation of the imaginary component). I claim that the metric $m=\frac{dx^2+dy^2}{y^2}=\frac{dz\,d\bar{z}}{\operatorname{Im}(z)^2}$ is G-invariant. This is a standard computation in coordinates:

$$\operatorname{Im}(gz)^2 = \left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)^2 = \left(\frac{(az+b)(c\bar{z}+d)-(a\bar{z}+b)(cz+d)}{|cz+d|^2}\right)^2$$

$$= \left(\frac{z(ad-bc)-\bar{z}(-bc+ad)}{|cz+d|^2}\right)^2 = \frac{\operatorname{Im}(z)^2}{|cz+d|^4}$$

$$dgz = \frac{a(cz+d) - c(az+b)}{(cz+d)^2}\, dz = \frac{dz}{(cz+d)^2}$$

$$d\overline{gz} = \frac{d\bar{z}}{(c\bar{z}+d)^2}$$

$$\implies \frac{dgz\, d\overline{gz}}{\mathrm{Im}(gz)^2} = \frac{|cz+d|^4\, dz\, d\bar{z}}{\mathrm{Im}(z)^2 (cz+d)^2 (c\bar{z}+d)^2} = \frac{dz\, d\bar{z}}{\mathrm{Im}(z)^2}$$

Now, because $\sqrt{|\det y^2 I|} = y^2$ (y is positive), the Riemann volume form of the invariant metric is given by $\frac{dx\, dy}{y^2}$. Note that it is also convenient to instead consider the action of $SL_2(\mathbb{R})$ on the unit disc model D of hyperbolic space by map $f : \mathcal{H} \to D$ by $f(z) = \frac{z-i}{z+i}$. The action is easiest to write by thinking of $SL_2(\mathbb{R})$ as a Lie group it is isomorphic to:

$$SU(1,1) = \{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \}$$

An explicit isomorphism can be given by $a = \mathrm{Re}(\alpha) - \mathrm{Im}(\beta), b = \mathrm{Re}(\beta) - \mathrm{Im}(\alpha), c = \mathrm{Re}(\beta) + \mathrm{Im}(\alpha), d = \mathrm{Re}(\alpha) + \mathrm{Im}(\beta)$. The action on D is given by

$$g \cdot w = \frac{\alpha w + \beta}{\bar{\beta} w + \bar{\alpha}}$$

A similar calculation as before shows that the metric $(1 - |w|^2)^{-2}\, dx\, dy$ is G-invariant and has volume form $\frac{dw\, d\bar{w}}{1 - \bar{w}w}$. Finally, note that by adding together our formulas from above, our Lie group isomorphism maps the rotation matrices to

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

§4.2 Principal Series Representations

Using this action, we can begin to construct the irreducible unitary representations of $SL_2(\mathbb{R})$. The subsequent list of families of irreducibles will be exhaustive, but I will not attempt to prove this. One of the few similarities between the $SL_2(\mathbb{R})$ case and the case of $SU(2)$ is a sort of reducibility of the regular representation into a "direct integral": without attempting to be too precise (the details can be found in [Deitmar]) we have:

Theorem 4.3 (Plancherel Theorem)

For $G = SL_2(\mathbb{R})$ (or any unimodular "Type I" Lie group), there is a unique measure μ on $\widehat{G}$ (equivalence classes of irreducible unitary representations of G) such that for $f \in L^1(G) \cap L^2(G)$,

$$||f||_2^2 = \int_{(\widehat{G},\pi)} ||\pi(f)||_{HS}^2\, d\mu(\pi)$$

where HS denotes the Hilbert-Schmidt norm. Moreover, as $G \times G$ representations we have

$$L^2(G) = \int_{\widehat{G}} \pi \otimes \pi^*\, d\mu(\pi)$$

As we integrate over the space of irreducibles, there are "discrete" points and ones which exist in a continuum, similar to the spectrum of an operator. The former are called discrete series representations and the latter are called principal series representations. I'll begin with a discussion of principal series representations, following [Knapp].

On $H = L^2(\mathbb{R})$, define representations $\mathcal{P}^{iv,+}, \mathcal{P}^{iv,-}$ for $v \in \mathbb{R} \setminus \{0\}$ by

$$\rho_{iv,+}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f(x) = |-bx+d|^{-1-iv} f\left(\frac{ax-c}{-bx+d}\right)$$

$$\rho_{iv,-}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f(x) = \operatorname{sgn}(-bx+d)|-bx+d|^{-1-iv} f\left(\frac{ax-c}{-bx+d}\right)$$

These representations are unitary because

$$\int_{\mathbb{R}} \overline{|-bx+d|^{-1-iv} f\left(\frac{ax-c}{-bx+d}\right)} |-bx+d|^{-1-iv} g\left(\frac{ax-c}{-bx+d}\right) dx =$$

$$= \int_{\mathbb{R}} f(y) g(y)\, dx$$

using change of variables (and a real raised to the power of a multiple of i is on the unit circle) with

$$\frac{d}{dx}\frac{ax-c}{-bx+d} = \frac{a(-bx+d)+b(ax-c)}{(-bx+d)^2} = \frac{1}{(-bx+d)^2}$$

and similarly for $\mathcal{P}^{iv,-}$. Let E be an orthogonal projection onto some closed invariant subspace of $L^2(\mathbb{R})$. We showed in Chapter 3 of this report that such an E commutes with the action of G. For any $y \in \mathbb{R}$,

$$\rho_{+,iv}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} f(x) = f(x-y)$$

and so E commutes with shifts. It is a fact in Fourier analysis that if an operator commutes with translations there exists an $m \in L^\infty(\mathbb{R})$ such that $(E\hat{f})(\xi) = m(\xi)\hat{f}(\xi)$ for any $f \in L^2(\mathbb{R})$ [Stein]. Then, $E^2 = E$ is a projection, so $m^2 = m$ i.e. m is 0 or 1 almost everywhere. Now, E also commutes with

$$\rho_{+,iv}\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} f(x) = |r|^{1+iv} f(r^2 x)$$

and so $m(x)\hat{f}(r^2x) = m(r^2x)\hat{f}(r^2x)$, meaning m is constant on the positive and negative half-lines. That means there are two nontrivial closed invariant subspaces: the space of functions whose Fourier transform is constant on the positive half-line and the space whose transform is constant on the negative half-line. The way to get a representation which is truly irreducible is to instead act on the space of densities $\varphi(x)(dx)^{(1+s)/2}$ which can be integrated against (pseudo-densities for $\mathcal{P}^{iv,-}$) [Gruson].

§4.3 Discrete Series Representations

The discrete portion of the direct integral is comprised of the discrete series representations, which are exactly the irreducible representations whose matrix coefficients are in $L^2(G)$ (as we had for all irreducibles of $SU(2)$) [Gruson]. Now, there is a general theorem that a simple Lie group has a holomorphic discrete series representation — one which forms some Hilbert space of holomorphic functions — iff it has a maximal compact subgroup with infinite center [Knapp]. In our case, the rotation matrices S^1 form a maximal compact subgroup of $SL_2(\mathbb{R})$, and the Hilbert space will come from holomorphic functions on $\mathcal{H}$.

For any positive integer n, define $\mathcal{H}_n^+$ as the space of holomorphic densities on $\mathcal{H}$. These are holomorphic functions ψ on $\mathcal{H}$, which we write formally as $\psi(z)(dz)^{n/2}$, satisfying the property that

$$\int_{\mathcal{H}} |\psi(z)|^2 y^{n-2}\, dz\, d\bar{z} < \infty$$

using the invariant metric we calculated in the last section. We can equip $\mathcal{H}_n^+$ with a Hilbert space structure via

$$\langle \psi(z)(dz)^{n/2}, \phi(z)(dz)^{n/2} \rangle := \int_{\mathcal{H}} \overline{\psi(z)}\phi(z) y^{\max(n-2,0)}\, dz\, d\bar{z}$$

which is well-defined by Hölder's inequality. We can obtain a representation of $SL_2(\mathbb{R})$ by the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\psi(z)(dz)^{n/2}\right) := \psi(\frac{az+b}{cz+d})\frac{1}{(cz+d)^n}(dz)^{n/2}$$

Now, $SL_2(\mathbb{R})$ acts by holomorphic maps on $\mathcal{H}$ and so this density is still holomorphic. Moreover, this is in fact a unitary representation because for $n > 1$ we can calculate

$$\langle \rho_g\left(\psi(z)(dz)^{n/2}\right), \rho_g\left(\phi(z)(dz)^{n/2}\right)\rangle = \int_{\mathcal{H}} \frac{\overline{\psi(g\cdot z)}}{\overline{(cz+d)^n}} \frac{\phi(g\cdot z)}{(cz+d)^n} \mathrm{Im}(z)^n \frac{dz\, d\bar{z}}{\mathrm{Im}(z)^2}$$

$$= \int_{\mathcal{H}} \frac{\overline{\psi(g\cdot z)}}{\overline{(cz+d)^n}} \frac{\phi(g\cdot z)}{(cz+d)^n} \mathrm{Im}(z)^n \frac{dgz\, d\overline{gz}}{\mathrm{Im}(gz)^2} = \int_{\mathcal{H}} \overline{\psi(g\cdot z)}\phi(g\cdot z)\mathrm{Im}(gz)^{n-2} \frac{dgz\, d\overline{gz}}{\mathrm{Im}(gz)^2} = \langle \psi, \phi \rangle$$

using the differentials we calculated earlier.

We now show that $\mathcal{H}_n^+$ is irreducible. When we shift our action of G to the action of $SU(1,1)$ on D, the action on holomorphic functions becomes

$$\left(\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot f\right)(w) = (-\bar{\beta}w + \alpha)^{-n} f\left(\frac{\bar{\alpha}w - \beta}{-\bar{\beta}w + \alpha}\right)$$

Because we assumed f is holomorphic, and so analytic, the densities $w^k(dw)^{n/2}$ for $k \geq 0$ form a topological basis. Moreover, these $w^k(dw)^{n/2}$ are orthogonal: if $k_1 < k_2$

$$\langle w^{k_1}, w^{k_2} \rangle = \int_D \bar{w}^{k_1} w^{k_2} (1-|w|^2)^{n-2}\, dx\, dy = \int_D w^{k_2-k_1}|w|^{2k_1}(1-|w|^2)^{n-2}\, dx\, dy = 0$$

by symmetry. Now, rotations act on these functions by

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cdot w^k = w \mapsto e^{-ni\theta}(\frac{e^{-i\theta}w}{e^{i\theta}})^k = e^{(-n-2k)i\theta}w^k$$

so they are eigenfunctions of the action of S^1. This means any closed G-invariant (and so S^1-invariant) subspace S must contain elements of our orthogonal basis. However, a general g acts on w^k by

$$(g\cdot w^k)(w) = (-\bar{\beta}w+\alpha)^{-n}\left(\frac{\bar{\alpha}w-\beta}{-\bar{\beta}w+\alpha}\right)^k = (-\bar{\beta}w+\alpha)^{-n-k}(\bar{\alpha}w-\beta)^k$$

$$= \left(\sum_{i=0} -\bar{\beta}^i(-1)^i \frac{(n+k+i)!(-\bar{\beta}w+\alpha)^{-n-k-i}}{(n+k)!i!} z^i\right)\left(\sum_{i=0}^{k} \binom{k}{i} (\bar{\alpha}w)^{k-i}(-\beta)^i\right)$$

It can be shown that these coefficients are all nonzero [Gruson] and so if S is nonzero it must be the entire space (Alternatively, this can be shown in a more computationally friendly way by developing further results on the action of the Lie algebra, which is the approach taken by [Knapp]).

It is true generally that if one nonzero matrix coefficient of a unitary representation of a modular group is in $L^2(G)$ then all are [Knapp]. Thus, to show the discrete series representations have matrix coefficients in $L^2(G)$, we'll demonstrate they are on the constant 1_D function.

$$\langle g \cdot 1_D, 1_D \rangle = \int_D (\bar{\beta} w + \alpha)^{-n} (1 - |w|^2)^{n-2} \, dx \, dy$$

$$= \int_D \left(\sum_{i=0}^{\infty} \binom{-n}{i} (\bar{\beta} w)^i \alpha^{-n-i} \right) (1 - |w|^2)^{n-2} \, dx \, dy = \int_D \alpha^{-n-0} (1 - |w|^2)^{n-2} \, dx \, dy$$

using the generalized binomial series and the fact that the higher-order terms go to 0. In particular, letting c_n be the remaining integral, we have

$$\langle g \cdot 1_D, 1_D \rangle = \alpha^{-n} c_n$$

The final trick is that we can use the action of g on the point 0 satisfies $g \cdot 0 = \frac{\beta}{\bar{\alpha}}$ and so

$$1 - |g \cdot 0|^2 = 1 - \frac{|\beta|^2}{|\alpha|^2} = \frac{|\alpha|^2 - |\beta|^2}{|\alpha|^2} = \frac{1}{|\alpha|^2} = |\alpha|^{-2}$$

which means

$$|\langle g \cdot 1_D, 1_D \rangle|^2 = |\alpha|^{-2n} c_n^2 = c_n^2 (1 - |g \cdot 0|^2)^n$$

Putting all of this together, we can use the G invariance of $(1 - |w|^2)^{-2} \, dx \, dy$ from the earlier calculations to obtain the L^2-norm of the matrix coefficient is

$$\int_G |\langle g \cdot 1_D, 1_D \rangle|^2 \, dg = \int_G c_n^2 (1 - |g \cdot 0|^2)^n \, dg = c_n^2 \int_D (1 - |z|^2)^{n-2} \, dx \, dy < \infty$$

§4.4 Complementary Series Representations

The final class of irreducible representations of $SL_2(\mathbb{R})$ are the ones which are not in the regular representation, and furthermore aren't tempered (there is some $\epsilon > 0$ such that there is no basis of the representation with matrix coefficients in $L^{2+\epsilon}(G)$) [Knapp]. These complementary series representations are parameterized by a real $s \in (0, 1)$ and can be written as densities of the form $\varphi(x)(dx)^{(1+s)/2}$. Acting similarly as the other families, the inner product we equip this space with is the following:

$$\langle \varphi, \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\varphi(x)} \psi(y) |x - y|^{s-1} \, dx \, dy$$

Proof that this is an exhaustive list of irreducible representations will remain outside of the scope of this paper. Further details can be found in, for instance, [Godement].

§5 Conclusion

For $SU(2)$, compactness allowed us to show that the classification of unitary representations is not much more than the standard finite-dimensional representation theory: every unitary irreducible is finite-dimensional and conversely every finite-dimensional representation is unitarizable. Furthermore, the Peter-Weyl Theorem tells us that the regular representation of $SU(2)$ decomposes as a topological direct sum of the irreducibles with multiplicity given by their dimension, and orthonormal characters.

Despite having the same finite-dimensional complex representations as $SU(2)$, the noncompactness of $SL_2(\mathbb{R})$ makes its unitary representation theory significantly less well-behaved. The finite-dimensional representations tell us little because no finite-dimensional representations are unitarizable. Instead of a discrete set of unitary irreducibles that we had for $SU(2)$, two of the families of unitary irreducibles of $SL_2(\mathbb{R})$ are over continuously-varying parameters. Finally, only two of the provided families arise from the regular representation.

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