Unitarizability of Irreducible Highest-Weight Modules of Some Infinite-Dimensional Lie Algebras

Jon Hillery

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This semester we have constructed and found various properties of irreducible highest-weight representations of symmetrizable Kac-Moody algebras. Such representations are of interest from a purely mathematical perspective and because highest-weight representations often arise in quantum field theory. Similarly, it's natural from a physical and mathematical perspective to ask when an algebra acts unitarily. In this project, I will examine when a symmetrizable Kac-Moody algebra or the Virasoro algebra (as well as the Heisenberg algebra) have unitary actions on their highest-weight representations. There are complete classifications in all cases. I have decided to devote a large portion of the project to going through all of the details of the classification given in Chapter 11 of Kac's book ([10]) because many of them come from portions of previous chapters we did not discuss in class.

§1 Compact Forms and Unitarizability

To motivate our definition of a unitarizable representation, let's first restrict to considering the case when $\mathfrak{g} = \mathfrak{g}(A)$ is a finite-dimensional complex simple Lie algebra. Recall that a real Lie algebra is called compact if its Killing form is negative definite. Due to a theorem of Cartan [1], every such $\mathfrak{g}$ has a unique (up to isomorphism) compact real form $\mathfrak{k}$, a real Lie subalgebra of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$$

which is given by the fixed points of the compact antilinear involution defined on the Chevalley generators $\{e_i, h_i, f_i\}_i$ by

$$\omega_0(e_i) = -f_i \quad \omega_0(f_i) = -e_i \quad \omega_0(h_i) = -h_i$$

Letting G be the connected, simply-connected complex Lie group associated to $\mathfrak{g}$, a question for which much study has been devoted is the classification of irreducible unitary representations of G and their relation to the algebraic structure of $\mathfrak{g}$ and maximal compact subgroup K (with Lie algebra $\mathfrak{k}$), since the theory for compact groups is straightforward. Irreducible unitary representations of G fall into a larger nice class of representations called **admissible representations**, which are representations V on which K acts unitarily and each irreducible unitary representation of K has finite multiplicity in V. It is a nontrivial result of Harish-Chandra that an irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module V is the set of K-finite vectors of an irreducible unitary representation of G if and only if V is unitarizable [2].

Elaborating further in [2], an **admissible $(\mathfrak{g}, \mathfrak{k})$-module** for finite-dimensional $\mathfrak{g}$ is a finitely-generated $\mathfrak{g}$ module on which $U(\mathfrak{k})$ acts locally nilpotently and such that for all

$\mathfrak{k}$-modules W, $\mathrm{Hom}_{\mathfrak{k}}(W,V)$ is finite-dimensional. The **K-finite** vectors of V are members of the algebraic direct sum

$$V^{\text{finite}} = \bigoplus_{U \text{ irred unitary rep of } K} V[U]$$

where $V[U]$ is the copies of U in V. Members of V^{finite} are precisely those v for which the span over all k of $k \cdot v$ is finite-dimensional. Finally, V is **unitarizable** if there exists a positive-definite Hermitian form on V which is infinitessimally unitary, i.e. $H(g \cdot x, y) = -H(x, \omega_0(g) \cdot y)$ for all $x, y \in V, g \in \mathfrak{g}$ (on the compact subalgebra this is just a differentiated version of unitarity). With this background in mind, our goal will be to extend this notion of unitarizability to Kac-Moody algebras and discuss which highest-weight representations are unitarizable.

§1.1 Compact Form: Kac-Moody Algebra

Chapter 2 of [10] gives a definition of the compact involution for a Kac-Moody algebra. Because the Cartan subalgebra is not spanned by simple coroots when not in the finite-dimensional simple case, slightly more care must be taken to correctly extend antilinearly. Define a **real realization** of a generalized Cartan matrix A as a triple $(\mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^\vee)$, where $\mathfrak{h}_{\mathbb{R}}$ is a real vector space of dimension $2n-l$, such that $(\mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, \Pi, \Pi^\vee)$ is a usual complex realization of A. The fact that the entries of A are integers means we can always find an essentially unique real realization. From here on we will assume that one has been chosen and denote by $\mathfrak{g}(A)_{\mathbb{R}}$ the real subalgebra of $\mathfrak{g}(A)$ generated by the e_i, f_i, and $\mathfrak{h}_{\mathbb{R}}$.

Define the **compact involution** ω_0 on $\mathfrak{g}(A)$ by extending antilinearly

$$\omega_0(e_i) = -f_i \quad \omega_0(f_i) = -e_i \quad \omega_0(h) = -h, \forall h \in \mathfrak{h}_{\mathbb{R}}$$

As in the finite-dimensional theory, the **compact form** $\mathfrak{k}(A)$ of $\mathfrak{g}(A)$ is defined as the real Lie algebra consisting of the set of fixed points of ω_0. Since ω_0 is an antilinear involution, $\mathfrak{k}(A) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k}(A) \oplus i\mathfrak{k}(A) = \mathfrak{g}(A)$.

Definition 1.1. A pair (V, H), where V is a $\mathfrak{g}(A)$-module and H is a Hermitian form on V, is called **contravariant** if $H(g \cdot x, y) = -H(x, \omega_0(g) \cdot y)$ for all $g \in \mathfrak{g}(A), x, y \in V$. If additionally H is positive-definite then (V, H) is **unitarizable**.

Our first example of a Hermitian form on a representation of an infinite-dimensional symmetrizable Kac-Moody algebra is on $\mathfrak{g}(A)$ itself with the adjoint representation. Define $(\cdot|\cdot)_0$ on $\mathfrak{g}(A)$ by

$$(x|y)_0 := -(\omega_0(x)|y) \text{ for } x, y \in \mathfrak{g}(A)$$

where $(\cdot|\cdot)$ is a usual standard bilinear form we have defined previously. Bilinearity of the standard form and antilinearity of ω_0 guarantee that $(\cdot|\cdot)_0$ is Hermitian. Recall the standard $(\cdot|\cdot)$ pairs $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ nondegenerately when $\alpha = -\beta$ and otherwise orthogonally. Therefore, since ω_0 exchanges $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, $(\cdot|\cdot)_0$ pairs $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ nondegenerately if $\alpha = \beta$ (including $\alpha = \beta = 0$) and otherwise orthogonally.

Finally, to verify that $(\cdot|\cdot)_0$ is contravariant, we can directly compute by splitting into real and imaginary parts that

$$([x, y]|z)_0 = -(\omega_0([x_1 + ix_2, y_1 + iy_2])|z) =$$

$$= -(\omega_0([x_1, y_1]) - i\omega_0([x_1, y_2]) - i\omega_0([x_2, y_1]) - \omega_0([x_2, y_2])|z) =$$

$$= (-[\omega_0(x_1), \omega_0(y_1)] + i[\omega_0(x_1), \omega_0(y_2)] + i[\omega_0(x_2), \omega_0(y_1)] + [\omega_0(x_2), \omega_0(y_2)]|z) =$$

$$= -([\omega_0(x_1), \omega_0(y_1)]|z)+i([\omega_0(x_1), \omega_0(y_2)]|z)+i([\omega_0(x_2), \omega_0(y_1)]|z)+([\omega_0(x_2), \omega_0(y_2)]|z) =$$

$$= (\omega_0(y_1)|[\omega_0(x_1), z])-i(\omega_0(y_2)|[\omega_0(x_1), z])-i(\omega_0(y_1)|[\omega_0(x_2), z])-(\omega_0(y_2)|[\omega_0(x_2), z]) =$$

$$= (\omega_0(y_1) - i\omega_0(y_2)|[\omega_0(x_1), z] - i[\omega_0(x_2), z]) = (\omega_0(y)|[\omega_0(x), z]) = \boxed{-(y, [\omega_0(x), z])_0}$$

Thus, $(\cdot|\cdot)_0$ is a first example of a contravariant Hermitian form for a Kac-Moody algebra. A form with these properties is also essentially unique: from any contravariant Hermitian form H we can recover an invariant bilinear form on $\mathfrak{g}$ by restricting to $\mathfrak{k}$ so that $H([g,x],y) = -H(x,[g,y])$ and extending complex linearly. We can recover H from this bilinear form exactly how we defined $(\cdot|\cdot)_0$. Since we have previously discussed that invariant bilinear forms with these properties are unique up to scaling, the Hermitian form must be too.

Note that when $\mathfrak{g}(A)$ is not finite-dimensional the adjoint representation will not be highest weight, so $(\cdot|\cdot)_0$ is still not one of the forms we are primarily concerned with. However, the construction will be useful.

§2 Unitarizability for Symmetrizable Kac-Moody Algebras

In this section, I'll show that with our current definitions, the unitarizable highest-weight modules of symmetrizable Kac-Moody algebras are precisely the integrable ones, and (up to a constant) uniquely so. I will go through these results rather carefully, following [10] Chapter 11, because they require some background from earlier chapters we didn't cover in class.

First, the following theorem will tell us that actually the main question will just be which contravariant forms are positive-definite.

Theorem 2.1

For any Λ, $L(\Lambda)$ has a unique (up to a constant) nondegenerate contravariant Hermitian form H. With respect to H, the weight spaces of $L(\Lambda)$ are orthogonal.

Proof. We'll begin by first constructing a bilinear form which is contravariant with respect to the Chevalley involution by recalling some of our discussion of lowest-weight modules. Recall that the contragredient representation $L(\Lambda)^*$ is defined as the dual vector space to $L(\Lambda)$ equipped with the action $\pi_\Lambda^{\text{contra}}$ defined by

$$\pi_\Lambda^{\text{contra}}(g)\lambda = \big(x \mapsto \lambda(-\pi_\Lambda(g)x)\big)$$

Define the submodule

$$L^*(\Lambda) := \bigoplus_{\lambda \le \Lambda} (L(\Lambda)_\lambda)^* \subset L(\Lambda)^*$$

This $L^*(\Lambda)$ is a lowest-weight module for weight $-\Lambda$ because for $0 \neq \lambda \in (L(\Lambda)_\Lambda)^*$,

$$\pi_\Lambda^{\text{contra}}(f_i)\lambda = \big(x \mapsto \lambda(-\pi_\Lambda(f_i)x) = 0\big) = 0$$

$$\pi_\Lambda^{\text{contra}}(h)\lambda = \Big(x \mapsto \lambda(-\pi_\Lambda(h)x) = \lambda(-\langle\Lambda, h\rangle x) = -\langle\Lambda, h\rangle\lambda(x)\Big) = -\langle\Lambda, h\rangle\lambda$$

$$U(\mathfrak{g}(A))\lambda = L^*(\Lambda)$$

where the last equality can be obtained by expanding any element of $(L(\Lambda)_{\lambda'})^*$ in terms of a basis and using the analogous property for $L(\Lambda)$ (recall, weight spaces in the category $\mathcal{O}$ are finite-dimensional). Furthermore, this representation is irreducible: if V is a submodule, its vanishing set in $L(\Lambda)$ must be a submodule (and hence 0 or $L(\Lambda)$) because

$$\phi(g \cdot x) = (\pi_\Lambda^{\text{contra}}(g)\phi)(x) = \phi'(x) = 0$$

Thus, there is a bijection (since dualizing again gives an irreducible highest-weight representation) between $\mathfrak{h}^*$ and irreducible lowest-weight representations via $\Lambda \leftrightarrow L^*(-\Lambda)$.

Define an action π_Λ^* on $L(\Lambda)$ by

$$\pi_\Lambda^*(g)x := \pi_\Lambda(\omega(g))x$$

This equips $L(\Lambda)$ itself with the structure of a lowest-weight module of weight $-\Lambda$:

$$\pi_\Lambda^*(f_i)v_\Lambda = \pi_\Lambda(-e_i)v_\Lambda = 0$$

$$\pi_\Lambda^*(h)v_\Lambda = \pi_\Lambda(-h)v_\Lambda = -\langle \Lambda, h\rangle v_\Lambda$$
$$U(\mathfrak{g}(A))v_\Lambda = \omega(U(\mathfrak{g}(A)))v_\Lambda = L(\Lambda)$$

which is irreducible because pre-composing the action with ω does not change whether a subset is a submodule. By the correspondence between $\mathfrak{h}^*$ and irreducible lowest-weight modules, there is an isomorphism $\Phi : (L(\Lambda), \pi_\Lambda^*) \to (L^*(\Lambda), \pi_\Lambda^{\text{contra}})$. Define bilinear form B on $L(\Lambda)$ by

$$\boxed{B(x, y) := \Phi(x)y}$$

This B is nondegenerate because Φ is an isomorphism and contravariant with respect to ω because

$$B(g \cdot x, y) = \Phi(\pi_\Lambda(g)x)y = \Phi(\pi_\Lambda^*(\omega(g))x)y = \pi_\Lambda^{\text{contra}}(\omega(g))\Phi(x)y =$$

$$= (\Phi(x))(-\pi_\Lambda(\omega(g))y) = -B(x, \omega(g) \cdot y)$$

If $x \in L(\Lambda)_\lambda, y \in L(\Lambda)_\mu$ for $\lambda \neq \mu$ then

$$B(h \cdot x, y) = -B(x, \omega(h) \cdot y) \implies B(\lambda(h)x, y) = -B(x, -\mu(h)y)$$

$$\implies B(x, y)\,(\lambda(h) - \mu(h)) = 0 \implies B(x, y) = 0$$

i.e. distinct weight spaces are orthogonal under B. We can use this fact to show that B, as a nondegenerate contravariant bilinear form, is unique up to a constant: any form B' induces a map $L(\Lambda) \to L^*(\Lambda)$. This B' is actually in this restricted dual $L^*(\Lambda)$ because for any $x \in L(\Lambda)$, $B(\omega_0(x), \cdot)$ has a nonzero action on only finitely-many weight spaces (and any representation in $\mathcal{O}$ has finite-dimensional weight spaces) by orthogonality of distinct weight spaces. Then, this map can be made into a $\mathfrak{g}(A)$-module morphism because

$$B'(g \cdot x, \cdot) = B'(x, \cdot) \circ (-\omega(g)\cdot)$$

so we choose an action defined similarly as π_Λ^*, making $L^*(\Lambda)$ a highest-weight module. This map is a $\mathfrak{g}(A)$-module isomorphism by nondegeneracy of B. Hence, to complete the proof of essential uniqueness of B we can use a form of Schur's Lemma: the only endomorphisms of any $L(\Lambda)$ are scalar multiples of the identity because highest-weight vectors must be sent to highest weight vectors. Uniqueness also immediately gives symmetry because defining B by $\Phi(y)x$ also gives such a form (and $B(x, x) = cB(x, x)$ implies the constant $c = 1$).

Using this construction, we can construct the desired Hermitian form on $L(\Lambda)$. Letting $L(\Lambda)_{\mathbb{R}} := U(\mathfrak{g}(A)_{\mathbb{R}})v_\Lambda$, which complexifies to $L(\Lambda)$, we can define H by restricting B to $L(\Lambda)_{\mathbb{R}}$ and then extending conjugate-linearly. In general, any real bilinear form can be extended antilinearly through tensoring or concretely as

$$H(x_1 + ix_2, y_1 + iy_2) := B_{\mathbb{R}}(x_1, y_1) - iB_{\mathbb{R}}(x_1, y_2) + iB_{\mathbb{R}}(x_2, y_1) + B_{\mathbb{R}}(x_2, y_2)$$

Since ω_0 agrees with ω on $\mathfrak{g}(A)_{\mathbb{R}}$, the restriction is ω_0-contravariant. By defining the form via antilinear extension just as we did for ω_0, the extension is ω_0-contravariant on all of $L(\Lambda)$. This H is also again unique up to a constant: any form must restrict to a real contravariant form that extends complex-bilinearly to a scalar multiple of our constructed B. Finally, orthogonality of weight spaces under B immediately gives that weight spaces are orthogonal under H. □

There is another way of defining B and H given in the earlier ninth chapter of [10] which can be useful for some computations and is helpful understanding the form. We know we will have orthogonal weight spaces, so the weight-space structure and the value on the one-dimensional highest-weight space should tell us the form. To do so, for a fixed highest-weight vector v_Λ, define the **expectation** $\langle v \rangle \in \mathbb{C}$ of $v \in L(\Lambda)$ as its v_Λ-component:

$$v = \langle v \rangle v_\Lambda + \sum_{\alpha \in Q_+ \setminus \{0\}} v_{\Lambda - \alpha}$$

Then B can be constructed as

$$\boxed{B(gv_\Lambda, g'v_\Lambda) = \langle \widehat{\omega}(g) g' v_\Lambda \rangle}$$

where $\widehat{\omega}$ is the multiplication-reversing extension of $-\omega$ to all of $U(\mathfrak{g}(A))$. This is ω-contravariant because

$$B(g_0 g v_\Lambda, g' v_\Lambda) = \langle \widehat{\omega}(g_0 g) g' v_\Lambda \rangle = \langle \widehat{\omega}(g) - \omega(g_0) g' v_\Lambda \rangle = B(g_0 g v_\Lambda, -\omega(g_0) g' v_\Lambda)$$

and symmetric because

$$B(gv_\Lambda, g'v_\Lambda) = \langle \widehat{\omega}(g) g' v_\Lambda \rangle = \langle \widehat{\omega}(\widehat{\omega}(g) g') v_\Lambda \rangle = \langle \widehat{\omega}(g') g v_\Lambda \rangle$$

from the definition of the expectation. Similarly, we can construct H as

$$\boxed{H(gv_\Lambda, g'v_\Lambda) = \langle \widehat{\omega}_0(g) g' v_\Lambda \rangle}$$

where $\widehat{\omega}_0$ is the antilinear multiplication-reversing extension of $-\omega_0$ to all of $U(\mathfrak{g}(A))$. Note that both have normalization $H(v_\Lambda, v_\Lambda) = 1 = B(v_\Lambda, v_\Lambda)$, which we will now assume to eliminate the constant factor.

With a construction and understanding of the contravariant form, we want to show that our constructed H is positive-definite exactly when $L(\Lambda)$ is integrable. Along the way, we will also prove and use that the earlier-constructed contravariant form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ (with the adjoint representation) is positive definite on $\mathfrak{n}_- + \mathfrak{n}_+$. However, it is positive-definite on the Cartan only in the finite-type case (which is a highest-weight integrable representation, thus unitarity here follows from our results). First we will need two lemmas. The content and proofs of both are similar to results we had in class in our computations of character formulas.

Lemma 2.2

Let $\Lambda \in P_+$ be a dominant integral weight and $\lambda, \mu \in P(\Lambda)$. Then,

1. $(\Lambda|\Lambda) - (\lambda|\mu) \geq 0$, with equality iff $\lambda = \mu \in W \cdot \Lambda$.
2. $|\Lambda + \rho|^2 - |\lambda + \rho^2| \geq 0$ with equality iff $\lambda = \Lambda$ (recall ρ is defined by $\langle \rho, \alpha_i^\vee \rangle = \frac{1}{2} a_{ii}$).

Proof. 1. First, assume that $\lambda \in P_+$. Define $\beta := \Lambda - \lambda, \beta_1 := \Lambda - \mu$. Then,

$$(\Lambda|\beta) + (\lambda|\beta_1) = (\Lambda|\Lambda) - (\Lambda|\lambda) + (\lambda|\Lambda) - (\lambda|\mu) = (\Lambda|\Lambda) - (\lambda|\mu)$$

Since Λ is a highest weight, we know $\beta, \beta_1 \in Q_+$. Therefore, for some nonnegative integers k_i,

$$(\Lambda|\beta) = \sum_i k_i (\Lambda|\alpha_i) = \sum_i \frac{k_i}{\epsilon_i} \langle \Lambda, \alpha_i^\vee \rangle \geq 0$$

because Λ is dominant integral. The same argument shows that $(\lambda|\beta_1) \geq 0$, using the assumption $\lambda \in P_+$. Thus, $(\Lambda|\Lambda) - (\lambda|\mu) \geq 0$ with equality iff $(\Lambda|\beta) = (\lambda|\beta_1) = 0$. In this case, $\langle \Lambda, \alpha_i^\vee \rangle = 0$ for all $i \in \operatorname{supp}(\beta)$.

Suppose $\lambda \neq \Lambda$, i.e. this support is nonempty, and let S be a connected component of $\operatorname{supp}(\Lambda - \lambda)$ and $\mathfrak{n}_-(S) \subset \mathfrak{n}_-$ the algebra generated by $\{f_i\}_{i \in S}$. By definition, we have that

$$L(\Lambda)_\lambda \subset U(\mathfrak{n}_-)\mathfrak{n}_-(S)L(\Lambda)_\lambda$$

In particular, there exists some $i \in S$ such that $\langle \Lambda, \alpha_i^\vee \rangle \neq 0$ in order for $L(\Lambda)_\lambda \neq 0$, giving a contradiction. Thus, $\lambda = \Lambda$. However, with $\lambda = \Lambda$ the same argument gives that $(\lambda|\beta_1) = (\Lambda|\beta_1) = 0$ iff $\mu = \Lambda$. So, under our working assumption that $\lambda \in P_+$, we have equality in (1) iff $\lambda = \Lambda = \mu$.

Our assumption that $\lambda \in P_+$ was unnecessary due to invariance of $(\cdot|\cdot)$ and $P(\Lambda)$ under the Weyl group: for any $\lambda \in P(\Lambda)$ we have previously shown there is a unique $\lambda_+ \in P_+ \cap P(\Lambda)$ (minimizing $\operatorname{ht}(\Lambda - \lambda_+)$) such that $\lambda = w\lambda_+$ for some $w \in W$, and so

$$(\Lambda|\Lambda) - (\lambda|\mu) = (\Lambda|\Lambda) - (w\lambda_+|w(w^{-1}\mu)) = (\Lambda|\Lambda) - (\lambda_+|w^{-1}\mu) \geq 0$$

with equality iff $\lambda_+ = w^{-1}\mu = \Lambda$, i.e. $\lambda = \mu \in W \cdot \Lambda$.

2. We can re-write

$$|\Lambda + \rho|^2 - |\lambda + \rho|^2 = (\Lambda + \rho|\Lambda + \rho) - (\lambda + \rho|\lambda + \rho) = ((\Lambda|\Lambda) - (\lambda|\lambda)) + 2(\Lambda - \lambda|\rho)$$

We just showed that the first term is nonnegative. The second term is given by

$$2(\Lambda - \lambda|\rho) = 2\sum_i k_i(\alpha_i|\rho) = \sum_i k_i(\alpha_i|\alpha_i) \geq 0$$

Equality means both terms are 0 and in particular each $k_i = 0$, and so $\Lambda = \lambda$. □

We have previously defined one form of 'partial' Casimir operator by

$$\Omega_0 := 2 \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)}$$

following Kac's notation where for each positive root α, $\{e_\alpha^{(i)}\}_i$ is a fixed basis of the root space $\mathfrak{g}_\alpha$ and $\{e_{-\alpha}^{(i)}\}_i$ is the dual basis with respect to our standard form. This has a well-defined action on any $\mathfrak{g}(A)$-module V on which only finitely-many positive root spaces have a nonzero action on any fixed vector $v \in V$, called a **restricted** representation (in particular Ω_0 acts on every representation from the category $\mathcal{O}$). It was used in the discussion of an operator Ω that always commutes with the $\mathfrak{g}(A)$-action on such a V.

We now define a similar operator acting on $x \in \mathfrak{n}_-$ as follows:

$$\Omega_1(x) := \sum_{\alpha\in\Delta_+}\sum_i [e^{(i)}_{-\alpha}, [e^{(i)}_{\alpha}, x]_-]$$

where the subscript $-$ denotes taking the $\mathfrak{n}_-$-component. This operator is well-defined because for any $x \in \mathfrak{g}_{-\beta}$, we know $\beta \in \Delta_+$, $[e^{(i)}_\alpha, x] \in \mathfrak{g}_{\alpha-\beta}$ which only has possibly nonzero $\mathfrak{n}_-$-component if each α_i-component of α is less than the height of β. This Ω_1 will be useful in further computations with Ω and Ω_0.

Lemma 2.3

For $\alpha \in \Delta_+, x \in \mathfrak{g}_{-\alpha}$,

$$\Omega_1(x) = (2(\rho|\alpha) - (\alpha|\alpha))x$$

Proof. Let $M(0) = U(\mathfrak{n}_-)v$ be the Verma module of highest weight 0 with generating highest-weight vector v. The proof will be by two different computations of $\Omega_0 x(v)$ in $M(0)$. Since $M(0)$ is in $\mathcal{O}$, we do know that $\Omega_0(w)$ is well-defined for any $w \in M(0)$ and in particular for $w = xv$, so $\Omega_0 x(v)$ is well-defined. By definition,

$$\Omega_0 x(v) = 2\sum_{\beta\in\Delta_+}\sum_i e^{(i)}_{-\beta} e^{(i)}_{\beta} x(v)$$

Since v is a highest-weight vector for $M(0)$, meaning $e^{(i)}_\beta$ acts by 0 on v,

$$\Omega_0 x(v) = 2\sum_{\beta\in\Delta_+}\sum_i e^{(i)}_{-\beta}[e^{(i)}_{\beta}, x](v)$$

Define $S := \{\beta \in \Delta_+ : \beta < \alpha\}$, allowing us to look at a finite set where this expression will be nonzero. Again using that v is a highest-weight vector of weight 0 and that bracketing acts additively on root spaces,

$$\Omega_0 x(v) = 2\sum_{\beta\in S}\sum_i e^{(i)}_{-\beta}[e^{(i)}_{\beta}, x](v) = 2\sum_{\beta\in S}\sum_i \left([e^{(i)}_{-\beta}, [e^{(i)}_{\beta}, x]] + [e^{(i)}_{\beta}, x]e^{(i)}_{-\beta}\right)(v) =$$

$$= \sum_{\beta\in S}\sum_i \left([e^{(i)}_{-\beta}, [e^{(i)}_{\beta}, x]] + [e^{(i)}_{\beta}, x]e^{(i)}_{-\beta}\right)(v) + \sum_{\beta\in S}\sum_i e^{(i)}_{-\beta}[e^{(i)}_{\beta}, x](v) =$$

$$= \sum_{\beta\in S}\sum_i \left([e^{(i)}_{-\beta}, [e^{(i)}_{\beta}, x]] + [e^{(i)}_{\beta}, x]e^{(i)}_{-\beta} + e^{(i)}_{-\beta}[e^{(i)}_{\beta}, x]\right)(v) =$$

$$= \boxed{\sum_{\beta\in S}\sum_i [e^{(i)}_{-\beta}, [e^{(i)}_{\beta}, x]](v)}$$

The last equality comes from the following identity:

$$\sum_i [e^{(i)}_{\beta}, x]e^{(i)}_{-\beta} + e^{(i)}_{-\beta}[e^{(i)}_{\beta}, x] = 0$$

We can prove this identity as follows: extend our standard form $(\cdot|\cdot)$ to $\mathfrak{g}(A) \otimes_{\mathbb{C}} \mathfrak{g}(A)$ via

$$(x \otimes y|x' \otimes y') := (x|x')(y|y')$$

For arbitrary $\alpha', \beta' \in \Delta$ and z, we have that for any $e \in \mathfrak{g}_{\alpha'}, f \in \mathfrak{g}_{\beta'}$,

$$\sum_i (e^{(i)}_{-\alpha'} \otimes [z, e^{(i)}_{\alpha'}]|e \otimes f) = \sum_i (e^{(i)}_{-\alpha'}|e)([z, e^{(i)}_{\alpha'}]|f) = -\sum_i (e^{(i)}_{-\alpha'}|e)(e^{(i)}_{\alpha'}|[z, f]) =$$

$$= -\sum_i (e^{(i)}_{-\alpha'}|\sum_k c_k e^{(k)}_{\alpha'})(e^{(i)}_{\alpha'}|[z, f]) = -\sum_i c_i(e^{(i)}_{\alpha'}|[z, f]) = (e|[f, z]) =$$

$$= ([z, e]|f) = \sum_i (e^{(i)}_{-\beta'}|[z, e])(e^{(i)}_{\beta'}|f) = \sum_i ([e^{(i)}_{-\beta'}, z] \otimes e^{(i)}_{\beta'}|e \otimes f)$$

It is straightforward to check that nondegeneracy of bilinear forms extends to the tensor product for finite-dimensional spaces (like our root spaces are), and so we have for every such z that

$$\sum_i e^{(i)}_{-\alpha'} \otimes [z, e^{(i)}_{\alpha'}] = \sum_i [e^{(i)}_{-\beta'}, z] \otimes e^{(i)}_{\beta'}$$

In our case, viewing $\mathfrak{g}(A) \otimes_{\mathbb{C}} \mathfrak{g}(A) \subset U(\mathfrak{g}(A))$, and setting $z = x, \alpha' = \beta, \beta' = -\beta$, we have

$$\sum_i e^{(i)}_{-\alpha'}[z, e^{(i)}_{\alpha'}] = -\sum_i [z, e^{(i)}_{-\beta'}]e^{(i)}_{\beta'}$$

as desired.

Now, let's calculate $\Omega_0 x(v)$ in a different way. The formula we want to obtain is

$$\boxed{\Omega_0 x(v) = (2(\rho|\alpha) - (\alpha|\alpha))x(v)}$$

We have previously discussed in class that $\Omega = 2\nu^{-1}(\rho) + \sum_i u^i u_i + \Omega_0$ (the u are dual bases of $\mathfrak{h}$ with the standard form) commutes with the action of $\mathfrak{g}(A)$ and acts on $M(0)$ by $(0 + 2\rho|0)\mathrm{Id} = 0$. Therefore,

$$\Omega_0 x(v) = -2\nu^{-1}(\rho)x(v) - \sum_i u^i u_i x(v) =$$

$$= -2\langle \nu^{-1}(\rho), -\alpha\rangle x(v) - \sum_i \langle u^i, -\alpha\rangle\langle u_i, -\alpha\rangle x(v) = -2\langle \nu^{-1}(\rho), -\alpha\rangle x(v) =$$

$$= 2(\rho|\alpha)x(v) - \sum_i (u^i|\nu^{-1}(-\alpha))(u_i|\nu^{-1}(-\alpha))x(v) = 2(\rho|\alpha)x(v) - (\nu^{-1}(-\alpha)|\nu^{-1}(-\alpha))x(v) =$$

$$= (2(\rho|\alpha) - (\alpha|\alpha))x(v)$$

Combining the two boxed formulas, we have

$$\sum_{\beta \in S}\sum_i [e^{(i)}_{-\beta}, [e^{(i)}_{\beta}, x]](v) = (2(\rho|\alpha) - (\alpha|\alpha))x(v)$$

Note that the coefficients on both sides are in $U(\mathfrak{n}_-)$ because x is. Therefore, since the Verma module $M(0)$ was defined as a free $U(\mathfrak{n}_-)$-module, we have

$$\boxed{\sum_{\beta \in S}\sum_i [e^{(i)}_{-\beta}, [e^{(i)}_{\beta}, x]] = (2(\rho|\alpha) - (\alpha|\alpha))x}$$

□

We are now ready for the main theorem.

Theorem 2.4

For any symmetrizable Kac-Moody algebra $\mathfrak{g}(A)$,

1. When restricted to any root space $\mathfrak{g}_\alpha$ (but not $\mathfrak{h}$), $(\cdot|\cdot)_0$ is positive-definite. In particular, $(\cdot|\cdot)_0$ is positive-definite on $\mathfrak{n}_- \oplus \mathfrak{n}_+$.
2. Highest-weight module $L(\Lambda)$ is unitarizable iff $\Lambda \in P_+$. In particular, integrable highest-weight modules are unitarizable.

Proof. 1. We will show that $(\cdot|\cdot)_0$ is positive-definite on $\mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta_+$. This is sufficient because for the positive root spaces we can use that

$$(\omega_0(x)|\omega_0(x))_0 = -(x|\omega_0(x)) = -(\omega_0(x)|x) = (x|x)_0$$

Proceed by induction on the height of α. In the case $\mathrm{ht}(\alpha) = 1$, $\alpha = \alpha_i$ and so $\mathfrak{g}_{-\alpha} = \mathbb{C}f_i$. Therefore, the base case follows by definition of $(\cdot|\cdot)_0$:

$$(f_i|f_i)_0 = -(-e_i|f_i) = \epsilon_i > 0$$

Now, suppose $\mathrm{ht}(\alpha) = k$. Again set $S = \{\beta \in \Delta_+ : \beta < \alpha\}$. Each $\beta \in S$ has rank less than k and so by induction $(\cdot|\cdot)_0$ is positive-define on $\mathfrak{g}_{-\beta}$, so we can choose an orthonormal basis $\{e^{(i)}_{-\beta}\}_i$ for it. Defining $e^{(i)}_\beta := -\omega_0(e^{(i)}_{-\beta})$,

$$(e^{(i)}_\beta|e^{(j)}_{-\beta}) = (-\omega_0(e^{(i)}_{-\beta})|e^{(j)}_{-\beta}) = (e^{(i)}_{-\beta}|e^{(j)}_{-\beta})_0 = \delta_{i,j}$$

so these choices do provide a dual basis under $(\cdot|\cdot)$. By the previous lemma (2.3), for any $x \in \mathfrak{g}_{-\alpha}$,

$$((2(\rho|\alpha) - (\alpha|\alpha))x|x)_0 = (\Omega_1(x)|x)_0 = \sum_{\alpha' \in \Delta_+} \sum_i ([e'^{(i)}_{-\alpha'}[e'^{(i)}_{\alpha'}, x]_-]|x)_0 =$$

$$= \sum_{\beta \in S} \sum_i ([e^{(i)}_{-\beta}, [e^{(i)}_\beta, x]]|x)_0 = \sum_{\beta \in S} \sum_i -([e^{(i)}_\beta, x]|[\omega_0(e^{(i)}_{-\beta}), x])_0 =$$

$$= \sum_{\beta \in S} \sum_i ([e^{(i)}_\beta, x]|[e^{(i)}_\beta, x])_0 \geq 0$$

by induction since $[e^{(i)}_\beta, x] \in \mathfrak{g}_{\beta-\alpha}$. With the fact (which we will shortly prove) that for any $\alpha \in \Delta_+ \setminus \Pi$,

$$\boxed{2(\rho|\alpha) - (\alpha|\alpha) \geq 0}$$

we can divide to obtain

$$(x|x)_0 \geq 0 \ \forall x \in \mathfrak{g}_{-\alpha}$$

By non-degeneracy, $(x|x)_0 = 0$ only if $x = 0$, so $(\cdot|\cdot)_0$ is non-degenerate as desired.

Now it remains to show the boxed inequality for α above. For $\alpha \in \Delta_+$ and not a simple root, recall that α is called **real** if it is in the Weyl group orbit of a simple root and otherwise it is called **imaginary**. First suppose $\alpha = w(\alpha_i)$ is real. Then $\alpha^\vee = w(\alpha_i^\vee)$ is a real coroot of 'height' greater than 1. We know that

$$(\alpha|\alpha) = (w(\alpha_i)|w(\alpha_i)) = (\alpha_i|\alpha_i) > 0$$

and so

$$\frac{2(\rho|\alpha)}{(\alpha|\alpha)} = \langle \rho, \alpha^\vee \rangle = \sum_i k_i \frac{1}{2} \cdot 2 = \mathrm{ht}(\alpha^\vee) > 1$$

Thus, the boxed inequality is true for real α.

Now, suppose α is imaginary. Since $(\rho|\alpha) = \sum_i \frac{k_i}{2}(\alpha_i|\alpha_i) > 0$ for all positive α, it suffices to show that for positive imaginary α we have that $(\alpha|\alpha) \leq 0$. Choose $\beta \in W \cdot \alpha$ in Δ_+ of minimal height. This β must have $\langle \beta, \alpha_i^\vee \rangle \leq 0$ for all i because it lies in $-C^\vee$ where $C^\vee$ is the co-fundamental chamber of the Weyl group action (by similar considerations as our construction of the fundamental chamber). Then,

$$(\alpha|\alpha) = (\beta|\beta) = \sum_i k_i(\beta|\alpha_i) \leq 0$$

completing the proof.

2. Similarly to the last part, we will show that H is positive-definite on each weight space $L(\Lambda)_\lambda$ by induction on $\mathrm{ht}(\Lambda - \lambda)$. In the base case we have $\lambda = \Lambda$ in which case $L(\Lambda)_\Lambda = \mathbb{C}v_\Lambda$ with $H(v_\Lambda, v_\Lambda) = 1$.

Suppose $\Lambda - \lambda$ has height $k > 0$ and let $v \in L(\Lambda)_\lambda$. As in the first part, we can choose orthonormal basis (with respect to $(\cdot|\cdot)_0$) $\{e_\alpha^{(i)}\}_i$ for $\mathfrak{g}_\alpha$ with $\alpha \in \Delta_+$ which will have dual basis $\{-\omega_0(e_\alpha^{(i)})\}$ for $\mathfrak{g}_{-\alpha}$ with respect to $(\cdot|\cdot)$. Recall that we have previously constructed operator

$$\Omega := 2\nu^{-1}(\rho) + \sum_i u^i u_i + \Omega_0$$

, where $\{u_i\}_i, \{u^i\}_i$ are dual bases of $\mathfrak{h}$, which acts on any restricted $\mathfrak{g}(A)$-module and commutes with the action of $\mathfrak{g}(A)$. Using our chosen dual bases, we can write Ω as

$$\Omega = 2\nu^{-1}(\rho) + \sum_i u^i u_i - 2\sum_{\alpha\in\Delta_+}\sum_i \omega_0(e_\alpha^{(i)})e_\alpha^{(i)}$$

This acts on $v \in L(\Lambda)_\lambda$ as

$$\Omega(v) = 2\langle \lambda, \nu^{-1}(\rho)\rangle v + \sum_i \langle \lambda, u^i\rangle\langle \lambda, u_i\rangle v - 2\sum_{\alpha\in\Delta_+}\sum_i \omega_0(e_\alpha^{(i)})e_\alpha^{(i)}(v) =$$

$$= (2\lambda|\rho)v + \sum_i (\nu^{-1}(\lambda)|u^i)(\nu^{-1}(\lambda), u_i)v - 2\sum_{\alpha\in\Delta_+}\sum_i \omega_0(e_\alpha^{(i)})e_\alpha^{(i)}(v) =$$

$$= (\lambda|2\rho + \lambda)v - 2\sum_{\alpha\in\Delta_+}\sum_i \omega_0(e_\alpha^{(i)})e_\alpha^{(i)}(v)$$

by the dual basis identity obtained in the proof of our last lemma (2.3). From this expression, we can compute $H(\Omega(v), v)$ in two different ways. First, we showed in class that Ω acts on $L(\Lambda)$ as

$$\Omega = (\Lambda + 2\rho|\Lambda)\mathrm{Id} \implies \boxed{H(\Omega(v), v) = (\Lambda + 2\rho|\Lambda)H(v, v)}$$

Secondly, the previous calculation gives directly that

$$H(\Omega(v), v) = (\lambda|2\rho + \lambda)H(v, v) - 2\sum_{\alpha\in\Delta_+}\sum_i H(\omega_0(e_\alpha^{(i)})e_\alpha^{(i)}(v), v) =$$

$$= \boxed{(\lambda|2\rho + \lambda)H(v, v) + 2\sum_{\alpha\in\Delta_+}\sum_i H(e_\alpha^{(i)}(v), e_\alpha^{(i)}(v))}$$

Combining these two gives

$$2\sum_{\alpha\in\Delta_+}\sum_i H(e_\alpha^{(i)}(v), e_\alpha^{(i)}(v)) = ((\Lambda + 2\rho|\Lambda) - (\lambda|2\rho + \lambda))\, H(v, v) =$$

$$= ((\Lambda|\Lambda) - (\lambda|\lambda) + 2(\Lambda - \lambda|\rho))\, H(v,v) = \left(|\Lambda + \rho|^2 - |\lambda + \rho|^2\right) H(v,v)$$

We showed that the coefficient $|\Lambda + \rho|^2 - |\lambda + \rho|^2$ is strictly positive for $\lambda < \Lambda$ in our second lemma (2.2) (we are using here that $\Lambda \in P^+$). Because we are taking $\alpha \in \Delta_+$, we know $e_\alpha^{(i)} v$ lies in some $L(\Lambda)_{\lambda'}$ with $\mathrm{ht}(\Lambda - \lambda') < k$, and so by induction the leftmost side of the equality is greater than or equal to 0. Dividing, we get $H(v,v) \geq 0$ and so (by nondegeneracy on weight spaces) H is positive-definite for $\Lambda \in P^+$.

Conversely, suppose that we know H is positive-definite on an arbitrary $L(\Lambda)$. Recall that $\{e_i, f_i, \alpha_i^\vee\}$ form an $\mathfrak{sl}_2$-subalgebra with the usual relations

$$[e_i, f_i^k] = -k(k-1)f_i^{k-1} + kf^{k-1}\alpha_i^\vee$$

$$\implies e_i f_i^k v_\Lambda = k(\Lambda(\alpha_i^\vee) - k + 1) f_i^{k-1} v_\Lambda$$

$$\implies 0 \leq H(f_i^k v_\Lambda, f_i^k v_\Lambda) = H(f_i^{k-1} v_\Lambda, -(-e_i) f_i^k v_\Lambda) =$$

$$= k(\langle \Lambda, \alpha_i^\vee \rangle - k + 1) H(f_i^{k-1} v_\Lambda, f_i^{k-1} v_\Lambda) = \cdots = k! \prod_{j=1}^{k} (\langle \Lambda, \alpha_i^\vee \rangle - j + 1) H(v_\Lambda, v_\Lambda)$$

Since $1 - j \leq 0$ for all $j \geq 1$ and $H(v_\Lambda, v_\Lambda) = 1$, that means $\langle \Lambda, \alpha_i^\vee \rangle \geq 0$ and so $\Lambda \in P^+$. □

§2.1 Untwisted Affine Case

With this classification completed, it is helpful to look at what this means for the case $\mathfrak{g}(A)$ is an untwisted affine Kac-Moody algebra where we have very concrete formulas. As previously, let $\mathring{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra, with

$$\widehat{\mathfrak{g}} = \mathring{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}K$$

$$\mathfrak{g} = \widehat{\mathfrak{g}} \oplus \mathbb{C}d$$

$$[x \otimes t^k, y \otimes t^l] = [x,y] \otimes t^{k+l} + k\delta_{k,-l}(x|y)K, \quad [d, x \otimes t^k] = kx \otimes t^k, \quad [K, \mathfrak{g}] = 0$$

We have also discussed previously some of the corresponding structure of these algebras. The Cartan subalgebra is given by

$$\mathfrak{h} = \mathring{\mathfrak{h}} + \mathbb{C}K + \mathbb{C}d$$

and we can extend a normalized invariant form on $\mathring{\mathfrak{g}}$ to $\mathfrak{g}$ via

$$(t^m \otimes x | t^n \otimes y) = \delta_{m,-n}(x|y), \quad (K|K) = (d|d) = 0$$

$$(K|d) = 1 \quad (K|\mathfrak{g}[t, t^{-1}]) = (d|\mathfrak{g}[t, t^{-1}]) = 0$$

To write our involutions in concrete terms, it will also be necessary to have a description of the simple roots and Chevalley generators of these algebras. Letting $\{E_i, F_i\}_i$ be the Chevalley generators of $\mathring{\mathfrak{g}}$, it is shown in Chapter 7 of [10] that we can complete them to the Chevalley generators for $\mathfrak{g}$ by setting $e_i = 1 \otimes E_i$, $f_i = 1 \otimes F_i$ and adding in the additional e_0, f_0 defined by

$$e_0 = t \otimes E_0 \quad f_0 = t^{-1} \otimes F_0$$

$$E_0, F_0 \in \mathring{\mathfrak{g}} \text{ with } (F_0|\mathring{\omega}(F_0)) = \frac{-2}{(\theta|\theta)} \quad E_0 = -\mathring{\omega}(F_0)$$

where is the highest root of $\mathring{\mathfrak{g}}$ (the positive root whose sum with any simple root is not a root of $\mathring{\mathfrak{g}}$). Using these generators, we can find an explicit description of our involutions:

$$\omega(e_i) = -f_i \implies \omega(t \otimes E_0) = -t^{-1} \otimes F_0$$

$$\omega(h) = -h \implies \omega(K) = -K \;\; \omega(d) = -d$$

Thus, the full form of the involution is

$$\omega(t^m \otimes x + \lambda K + \mu d) = t^{-m} \otimes \mathring{\omega}(x) - \lambda K - \mu d$$

Similarly, for the compact involution we obtain can expand from these same values antilinearly to obtain for any $P \in \mathbb{C}[t, t^{-1}]$

$$\omega_0(P(t) \otimes x + \lambda K + \mu d) = \overline{P}(t^{-1}) \otimes \mathring{\omega}_0(x) - \overline{\lambda} K - \overline{\mu} d$$

where $\overline{P}$ is obtained from P by taking the complex conjugate of each coefficient of P. We can now expand the Hermitian form as

$$(t^m \otimes x + \lambda_1 K + \mu_1 d | t^n \otimes y + \lambda_2 K + \mu_2 d)_0 = -(t^{-m} \otimes \mathring{\omega}_0(x) - \overline{\lambda_1} K - \overline{\mu_1} d | t^n \otimes y + \lambda_2 K + \mu_2 d) =$$

$$= -\delta_{m,n}(\mathring{\omega}_0(x)|y) - \overline{\lambda_1}\mu_2 - \overline{\mu_1}\lambda_2 = \delta_{m,n}(x|y)_0 - \overline{\lambda_1}\mu_2 - \overline{\mu_1}\lambda_2$$

In particular, we can show directly that $(\cdot|\cdot)_0$ is positive-define away from the Cartan as we did in Theorem 2.4:

$$(P(t) \otimes x | P(t) \otimes x)_0 = (x|x)_0 \cdot \sum_j |c_j|^2$$

where the c_j are the coefficients of P. This does assume positive-definiteness of $(\cdot|\cdot)_0$ for finite-dimensional simple Lie algebras, which can be shown directly by extending the negative of the Killing form on the compact real form of $\mathring{\mathfrak{g}}$ (which is necessarily positive-definite by compactness) antilinearly.

Notice $(\cdot|\cdot)_0$ is still not positive-definite or positive-semidefinite on $\mathfrak{h}$. However, if we restrict to $\mathfrak{h}'$ the cross-terms disappear and so $(\cdot|\cdot)_0$ actually is positive-semidefinite on $\mathfrak{h}'$. In fact, it can be shown that $(\cdot|\cdot)_0$ is positive-definite on $\mathfrak{h}'$ iff A is finite type, and positive semidefinite on $\mathfrak{h}'$ iff A is affine type [10].

§3 A Generalization for Affine Kac-Moody Algebras

In addition to the classification of irreducible highest-weight modules which are unitarizable with respect to our usual structures, there is also a classification for a more general choice of Borel subalgebra and involution given in [8] and [8]. A general subset Δ'_+ of Δ will be called a **set of positive roots** if it is closed under addition and contains exactly one of α and $-\alpha$ for every $\alpha \in \Delta$ (besides 0, which it does not contain). Then, the Borel algebra associated to Δ'_+ is

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta'_+} \mathfrak{g}_\alpha$$

A general antilinear anti-involution ω is called **consistent** if it respects the grading on $\mathfrak{g}(A)$: $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. In particular, ω can be rescaled on simple root spaces such that $\omega(e_i) = \pm f_i$.

The Verma module construction can be generalized by using instead of a weight a general one-dimensional representation $\lambda : \mathfrak{b} \to \mathbb{C}$, defining

$$M(\lambda) = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{b}^\lambda$$

where $\mathfrak{b}^\lambda = \{x \in \mathfrak{b} : \lambda(x) = 0\}$. This is highest-weight in the sense that it is has a generating vector on which $\mathfrak{b}$ acts by λ. Then, this $M(\lambda)$ has unique contravariant Hermitian form given by

$$H(u, v) = \lambda(\pi(\omega(v)u))$$

where π is the projection onto $U(\mathfrak{b})$. This is essentially the same as the expectation construction given before and contravariance and uniqueness follow from the same argument. Note that in order for this to be Hermitian we need

$$\lambda(\pi(g)) = \overline{\lambda(\pi(\omega(g)))}$$

which we will call the reality condition and henceforth assume. In order to try to construct representations where this form is unitarizable, we work backwards by defining $L(\lambda) = L_{\mathfrak{b},\omega}(\lambda) := M(\lambda)/I(\lambda)$ where $I(\lambda)$ is the kernel of H. On this quotient H passes to a non-degenerate contravariant form, but as before it is not necessarily positive-definite. There are essentially three categories of $\omega, \mathfrak{b}, \lambda$ for which these $L(\lambda)$ are unitarizable:

The first class is those defined with the compact involution, standard Borel subalgebra, and $\lambda(e_i) = 0, \lambda(\alpha_i^\vee) \in \mathbb{Z}_+$. These are exactly the representations we have already classified as being unitarizable in the last section. These will be referred to as the **integrable** ones.

The next class is the **elementary** representations. Let $\mathring{\mathfrak{b}}$ be a Borel subalgebra of $\mathring{\mathfrak{g}}$, $\mathring{\omega}$ a consistent antilinear anti-involution of $\mathring{\mathfrak{g}}$, and $\{\lambda_i\}_{i=1,\dots,N}$ a finite set of $\mathring{\mathfrak{b}}$-representations such that $L_{\mathring{\mathfrak{b}},\mathring{\omega}}(\lambda_i)$ is a unitarizable representation of $\mathring{\mathfrak{g}}$. Note that the classification of these modules for finite-dimensional simple algebras is certainly nontrivial and given in [7]. Define

$$\omega(t^k \otimes x) = t^{-k} \otimes \mathring{\omega}(x)$$
$$\mathfrak{b} = \mathbb{C}[t, t^{-1}] \otimes \mathring{\mathfrak{b}}$$
$$\lambda(t^k \otimes b) = \sum_i z_i^k \lambda_i(b)$$

where $z_i \in S^1 \subset \mathbb{C}$ are some chosen constants, and the central extension elements are negated by the involution and act by 0 under λ. The involution was chosen exactly as we calculated it to be for the standard setup in the last section, so its properties follow from those of $\mathring{\omega}$ immediately. Since the representation acts by merely collapsing the loop components to norm-1 constants (unitary one-dimensional operators), it is true directly that

$$L_{\omega,\mathfrak{b}}(\lambda) = L_{\mathring{\mathfrak{b}},\mathring{\omega}}(\lambda_1) \otimes \cdots \otimes L_{\mathring{\mathfrak{b}},\mathring{\omega}}(\lambda_N)$$

is unitarizable.

The final class of unitarizable representations is the **exceptional** ones. They exist only in the case $\mathring{\mathfrak{g}} = \mathfrak{sl}_{l+1}$ and are constructed via

$$\omega((a_{i,j}(t))_{i,j}) = (\varepsilon_{i,j}\overline{a_{j,i}}(t))_{i,j}$$
$$\mathfrak{b} = \{(a_{i,j}(t))_{i,j} : a_{i,j}(t) = 0 \text{ if } i > j\} + \mathfrak{h}$$
$$\lambda_\mu((a_{i,j}(t))_{i,j}) = -\int_{S^1} a_{1,1}(e^{i\theta})\, d\mu(\theta)$$

where μ is a finite positive measure on S^1, $(a_{i,j}(z))_{i,j}$ denotes the matrix with entries $a_{i,j}(z) \in \mathbb{C}[t, t^{-1}]$ viewed as elements of the loop algebra (and again having the central elements act by 0), and $\varepsilon_{i,j} = -1$ if both i and j are not 1 and otherwise $\varepsilon_{i,j} = 1$. These irreducible representations are unitarizable and parameterized by choice of the measure μ. Positive-definiteness, and thus unitarizability, in this case are shown in [8].

The following complete classification was given in [9]:

Theorem 3.1

If $\mathfrak{g}$ is an affine Lie algebra, ω a consistent antilinear anti-involution, $\mathfrak{b}$ a Borel subalgebra, and $\lambda : \mathfrak{b} \to \mathbb{C}$ a one-dimensional representation satisfying the reality condition, then $L_{\mathfrak{b},\omega}(\lambda)$ is unitarizable iff it is either an integrable representation, an elementary representation, an exceptional representation, or the highest component (irreducible component with highest highest-weight) of the tensor product of an elementary and exceptional representations.

§4 Unitarizability for the Virasoro Algebra

Another infinite-dimensional Lie algebra we have discussed for which there are nice unitarizability results is the Virasoro algebra. Recall that the Virasoro algebra was defined on the basis $\{L_n\}_{n\in\mathbb{Z}} \cup \{c\}$ with commutation relation

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n,-m}c$$

where $\mathbb{C}c$ is a one-dimensional central subspace. This algebra naturally arises from the study of Kac-Moody algebras as the nontrivial central extension by c of the algebra of derivations on the loop algebra ($L_n = -t^{n+1}\frac{\partial}{\partial t}$). To simplify things, I will abuse notation and use c for both the central element in the algebra and specific scalars by which it acts on modules.

In this case, the triangular decomposition is defined as

$$\mathrm{Vir} = \mathrm{Vir}_- \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_+ = \bigoplus_{i<0} \mathbb{C}L_i \oplus (\mathbb{C}c \oplus L_0) \oplus \bigoplus_{j>0} \mathbb{C}L_i$$

A functional on Vir_0 just corresponds to two elements of $\mathbb{C}$, meaning a highest weight module of weight (c,h) is defined as a representation V of the Virasoro algebra such that there exists nonzero $v_{c,h} \in V$ with

$$L_i v_{c,h} = 0 \text{ for } i > 0, \;\; U(\mathrm{Vir}_-)v_{c,h} = V, \;\; L_0 v_{c,h} = h v_{c,h}, \;\; c v_{c,h} = c v_{c,h}$$

Similarly to the constructions we did in our discussion of vertex algebras, there is a unique Verma module $M(c,h)$ corresponding to weight (c,h) which can be constructed via induced representations. By Poincaré-Birkhoff-Witt, $M(c,h)$ has a basis given by $\{L_{-j_n} \dots L_{-j_1} v_{c,h} : 0 < j_1 \le j_2 \le \dots \le j_n\}$. For $n = 1$ we have that

$$L_0 L_{-j} v_{c,h} = L_{-j} L_0 v_{c,h} + [L_0, L_{-j}]v_{c,h} = (h+j)L_{-j}v_{c,h}$$

and so inductively we can see that L_0 acts by the multiplication by $h + \sum_i j_i$:

$$L_0 L_{-j_n} \dots L_{-j_1} v_{c,h} = L_{-j_n} L_0 L_{-j_{n-1}} \dots L_{-j_1} v_{c,h} + [L_0, L_{-j_n}] L_{-j_{n-1}} \dots L_{-j_1} v_{c,h} =$$

$$= \left(h + \sum_{i=1}^{n-1} j_i\right) L_{-j_n} \dots L_{-j_1} v_{c,h} + j_n L_{-j_n} \dots L_{-j_1} v_{c,h} = \left(h + \sum_{i=1}^{n} j_i\right) L_{-j_n} \dots L_{-j_1} v_{c,h}$$

This gives us decomposition of $M(c,h)$ as

$$M(c,h) = \bigoplus_{j\ge 0} M(c,h)_{h+j} = \bigoplus_{j \ge 0} \{L_{-j_n} \dots L_{-j_1} v_{c,h} : \sum_i j_i = j\}$$

As an aside, since the highest-weight space is one-dimensional, we get a dimension formula for $\dim M(c,h)_{h+j} = p(j)$ where p counts the number of integer partitions of j. Now, we discussed in class that for V an $\mathfrak{h}$-diagonalizable module with $\mathfrak{h}$ commutative, any submodule respects the grading by $\mathfrak{h}$, and so with $\mathfrak{h} = \mathrm{Vir}_0$, we know $M(c,h)$ has a unique irreducible quotient $L(c,h)$ (using the grading, proper submodules are exactly the ones not containing the highest weight, so sums of proper submodules are proper submodules). This $L(c,h)$ also inherits the grading.

Which of these irreducible highest-weight representations are unitarizable? First, the Chevalley and compact involutions on Vir are defined as the linear (respectively, anti-linear) extensions of $L_n \mapsto -L_{-n}, c \mapsto -c$, as in the Kac-Moody case, so unitarizability will mean the module V possesses a positive-definite form H with $H(v_{c,h}, v_{c,h}) = 1$ and

$$\boxed{H(L_n u, v) = H(u, L_{-n}, v)}, \quad \boxed{c, h \in \mathbb{R}}$$

for all $u, v \in V$ (sufficient since c acts a scalar; the condition is redundant for h). Existence and uniqueness of an ω-contravariant bilinear form B and an ω_0-contravariant Hermitian form H works the same as for symmetrizable Kac-Moody algebras: uniqueness follows because the induced map on the restricted dual is a Vir-module isomorphism (which must preserve the highest-weight space), and the forms can explicitly be constructed as

$$B(g v_{c,h}, g' v_{c,h}) = \langle \widehat{\omega}(g) g' v_{c,h} \rangle$$

$$H(g v_{c,h}, g' v_{c,h}) = \langle \widehat{\omega}_0(g) g' v_{c,h} \rangle$$

where $\langle v \rangle$ denotes the $v_{c,h}$-component of v and the hats denote the multiplication-reversing extension to the universal enveloping algebra.

With this background, unitarizability over the Virasoro algebra is reduced just like the Kac-Moody case to a question of positive-definiteness. Full results are more complicated than the Kac-Moody case, but there are some immediate restrictions on c, h sketched in [10]:

Theorem 4.1

For arbitrary c, h,

1. $L(c,h)$ unitarizable $\implies h \geq 0, c \geq 0$
2. $L(0,h)$ unitarizable $\implies h = 0$.
3. If V is any unitarizable L_0-diagonalizable Virasoro module with finite-dimensional L_0-eigenspaces and the set of L_0-eigenvalues $\sigma(L_0)$ is bounded below, then V is an orthogonal direct sum of unitarizable $L(c,h)$'s and $\sigma(L_0)$ is nonnegative.

Proof. 1.

$$0 \leq H(L_{-1} v_{c,h}, L_{-1} v_{c,h}) = H(v_{c,h}, L_1 L_{-1} v_{c,h}) =$$
$$= H(v_{c,h}, L_{-1} L_1 v_{c,h}) + H(v_{c,h}, [L_1, L_{-1}] v_{c,h}) = H(v_{c,h}, 2L_0 v_{c,h}) = 2h$$

and similarly

$$0 \leq H(L_{-j} v_{c,h}, L_{-j} v_{c,h}) = H(v_{c,h}, (2jL_0 + \frac{j^3 - j}{12} c) v_{c,h}) = 2jh + \frac{j^3 - j}{12} c$$

which has for sufficiently large j that $0 < 2jh << |\frac{j^3-j}{12} c|$, necessitating $c \geq 0$.

2. For any positive n, we can calculate that

$$H(L_{-2n}v_{c,h}, L_{-2n}v_{c,h}) = 4nh$$

$$H(L_{-n}^2 v_{c,h}, L_{-2n}v_{c,h}) = 6n^2h$$

$$H(L_{-n}^2 v_{c,h}, L_{-n}^2 v_{c,h}) = 8n^2h^2$$

Therefore, when restricted to the subspace $\mathbb{C}L_{-2n}v_{c,h} \oplus \mathbb{C}L_{-n}^2 v_{c,h}$, H as a quadratic form has determinant

$$0 \leq 4nh \cdot 8n^2h^2 - (6n^2h)^2 = 4n^3h^2(8h - 9n)$$

Since $8h - 9n < 0$ for sufficiently large positive n, this means $h = 0$.

3. Under H, V is the direct sum of any submodule and its orthogonal complement (which is also a submodule by the grading), so we can totally decompose V into irreducibles. By the grading, eigenvalues of L_0 being bounded below means that V is a highest-weight module, and so it decomposes into $L(c,h)$'s. On each $L(c,h)$, L_0 acts on weight spaces by $h + j$ for some nonnegative j, so using (1) the eigenvalues of L_0 are nonnegative on all of V. □

Complete results have been known since 1986 due to [11] and [5]. I will state the classification and give some ideas about how it is proved.

Theorem 4.2 (Unitarizable Highest-Weight Representations for the Virasoro)
$L(c,h)$ is unitarizable exactly when $c \geq 1, h \geq 0$, or the (c, h) corresponds to a **discrete series** representations:

$$c = 1 - \frac{6}{m(m+1)}, \quad h = h_{p,q}(c) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$$

where m is an integer greater than 1 and $p \in \{1, 2, \ldots, m-1\}, q \in \{1, 2, \ldots, p\}$.

The key idea to restrict the values where the form could be positive-definite is contained in the determinant calculation we just did for part (2) of the previous theorem. We know that $L(c,h)$ decomposes into orthogonal weight spaces $L(c,h)_{h+j}$ defined as the image in the quotient of $\{L_{-j_n} \ldots L_{-j_1}v_{c,h} : \sum_i j_i = j\} \subset M(c,h)$, and so we can consider positive-definiteness on each of these spaces by computing the determinant of H when restricted to it. Taking $M_j(c,h)$ to be the matrix of H on $L(c,h)_{h+j}$, it is possible to obtain the "**Kac Determinant formula**"

$$\det M_j(c,h) = C_j \prod_{k=1}^{j} \left(\prod_{\substack{p,q \in \mathbb{Z}_+ \\ pq = k}} h - h_{p,q}(c) \right)^{\pi(j-k)}$$

where C_j is a positive constant depending on j and π is the integer partition function. This was shown in the aforementioned Kac papers (using, among other things, character formulas similar to ones we discussed in class), and then used through its number-theoretic properties to show for values of c and h not given in the theorem there is some j such that $\det M_j(c,h) < 0$.

These papers also gave realizations of $L(c,h)$ and a unitary form for the allowed values of c, h. In the case where $c \geq 1, h \geq 0$, it was constructed using the relation of the Virasoro algebra to the Heisenberg algebra and corresponding Fock space representations we discussed in class. I will briefly discuss in the next section the unitarizability for the Heisenberg algebra, but here is how it was used for the Virasoro. Let $F = \mathbb{C}[x_1, x_2, \dots]$ be the free commutative algebra of countably many variables (the Fock space) and have the Virasoro algebra act on F by

$$L_0 = \frac{1}{2}\mu^2 \mathrm{Id} + \sum_{j>0} j x_j \frac{\partial}{\partial x_j}, \quad L_n = \frac{1}{2}\sum_{i\in\mathbb{Z}} a_{-i}a_{n+i}$$

where for $n_0 = \mu\mathrm{Id}$ and for $a > 0$

$$a_{-n} = \sqrt{n}x_n, \quad a_n = \sqrt{n}\frac{\partial}{\partial x_n}$$

and c acts by $c\mathrm{Id}$. This space can be equipped with a contravariant Hermitian form by declaring elements of the form $\frac{x_1^{k_1}}{\sqrt{k_1!}} \dots \frac{x_s^{k_s}}{\sqrt{k_s!}}$ to be orthonormal. Moreover, when split into irreducibles, the component generated by 1 in F is $L(1, \frac{\mu^2}{2})$, obtaining unitarizability for $c = 1, h \geq 0$, and by taking tensor products for $c \in \mathbb{Z}_{\geq 1}, h \geq 0$. Then, from this result Kac argued one can show directly that $L(c, h)$ is unitarizable for $c \geq 1, h \geq 0$ by a continuity argument on the determinant formula.

To construct the discrete series unitary representations, the representation theory of affine Lie algebras was used. Recall that for for affine algebra $\widehat{\mathfrak{g}}$, we defined in class

$$V_k(\mathfrak{g}) = \mathrm{Ind}_{\mathring{\mathfrak{g}}[t^{-1},t]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}} \mathbb{C}_k$$

with $\mathring{\mathfrak{g}}[t^{-1}, t]$ acting by 0 on $\mathbb{C}_k$ and $\mathbf{1}$ acting by k. On this vertex algebra we constructed Sugawara operators

$$S_n := \frac{1}{2}\sum_{l+m=n} : J_l^a J_{a,m} :$$

where $J_l^a = J^a \otimes t^l, J_{a,m} = J_a \otimes t^{-m}$, $\{J^a, J_a\}_{a=1,\dots,\dim\mathring{\mathfrak{g}}}$ dual bases of $\mathring{\mathfrak{g}}$ under the normalized form $(\cdot|\cdot)$. These operators satisfies the commutation relation $[S_n, J_m^a] = -m(k + h^\vee)J_{m+n}^a$ and so for $k \neq h^\vee$ there is an action of the Virasoro algebra by $L_n = \frac{1}{k+h^\vee}S_n$ with central charge $c_k^{\mathring{\mathfrak{g}}} = \frac{\dim\mathring{\mathfrak{g}}k}{k+h^\vee}$.

For more general unitarizable representation of $\widehat{\mathfrak{g}}$, the Sugawara operators act similarly, giving a way to obtain unitarizable representations of the Virasoro algebra. The method used in [5] to obtain all discrete series representations did this, in addition to combining these representations with ones obtained from subgroups H of compact groups G corresponding to $\mathring{\mathfrak{g}}$. Specifically all can be constructed just using the case $G = \mathrm{Sp}(m-1), H = \mathrm{Sp}(m-2) \times \mathrm{Sp}(1)$.

§4.1 The Heisenberg Algebra

To complete the discussion of unitarizability to the main set of infinite-dimensional Lie algebras we have discussed in class, I will briefly recall everything needed for the easy classification for the case of the Heisenberg algebra. Recall that this algebra has basis $\{b_i\}_{i\in\mathbb{Z}} \cup \{\mathbf{1}\}$ where $\mathbf{1}$ is central and $[b_n, b_m] = n\delta_{n,-m}\mathbf{1}$. We showed in class that its irreducible highest-weight modules can be realized concretely as acting on the Fock space $\mathbb{C}[b_{-1}, b_{-2}, \dots]$ via b_n acting by multiplication for $n < 0$, $b_n \mapsto \lambda n\frac{\partial}{\partial b_{-n}}$ for $n > 0$, $b_0 \mapsto 0$, and $\mathbf{1} \mapsto \kappa\mathrm{Id}$ for $\lambda, \kappa \in \mathbb{C}$. Repeating the construction of the contravariant form we've been using,

$$H(b_{-j_1} \cdot b_{-j_m}, b_{-k_1} \dots b_{-k_n}) = \langle(\lambda j_m \frac{\partial}{\partial b_{-j_m}}) \dots (\lambda j_1 \frac{\partial}{\partial b_{-j_1}})b_{-k_1} \dots b_{-k_n}\rangle$$

where $\langle \cdot \rangle$ denotes the constant term (since that is the highest-weight space). In particular, distinct monomials are orthogonal under H. Therefore, it is immediate that H is positive-definite and the representation unitarizable exactly when $\lambda \in \mathbb{R}_{>0}, \kappa \in \mathbb{R}$.

§5 Conclusion

Over the course of this project, I discussed a complete classification for unitarizable highest-weight irreducible modules over symmetrizable Kac-Moody algebras, as well as over the Virasoro and Heisenberg algebra. With respect to the usual Borel subalgebra, compact involution, and highest-weight modules, the irreducible modules over symmetrizable $\mathfrak{g}(A)$ which are unitarizable are exactly $L(\Lambda)$ where Λ is dominant integral, i.e. the integrable representations. If we restrict ourselves to when A is of affine type but allow for more general involutions and Borel subalgebras, the main new types of unitarizable highest-weight are parameterized by positive finite measures on the circle. Finally, the Virasoro algebra has irreducible highest-weight modules parameterized by complex numbers c, h which are unitarizable for real $c \geq 1, h \geq 0$ and the c, h appearing in the discrete series representations.

From a pure mathematical perspective, the unitarizability structure is useful for further understanding the structure of highest-weight modules, as is done in Chapter 11 of [10]. In [6] and [12] unitarizability of Kac-Moody algebra representations is used to obtain unitary representations of infinite-dimensional Lie groups associated to these algebras by passing to the Hilbert space completion of $L(\Lambda)$.

Moreover, the question of unitarizability is relevant to physicists. As outlined in [4], the Virasoro algebra and affine Kac-Moody algebras can be thought of as central extensions of the Lie algebras associated to the group of diffeomorphisms of the circle, and loop groups, respectively. These can in turn be thought of as coordinate and gauge transformations on S^1, playing a central role in 2-dimensional conformal quantum field theory. In physical theories, these operators tend to act unitarily and the spectrum of the action of the respective commuting subalgebras usually correspond to physical quantities which are bounded below, which is why highest-weight unitarizable representations are of particular importance.

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