Topological Quantum Computation

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§1 Introduction

The limitations of our classical models and approaches to quantum computation are evident: it is very resource-intensive to correct for hardware-level errors and decoherence. Topological quantum computation is an effort to correct these errors at a hardware level by encoding quantum states in a way that is immune to local excitations or changes, i.e. a *topological* system. This is accomplished by *braiding* particles known as (nonabelian) *anyons*.

In traditional quantum mechanics in three dimensions, when a system is altered by only swapping two indistinguishable particles, the wavefunction of the resulting system is multiplied by either $+1$ or -1. In the first case the particles are known as bosons, and in the second case they are known as fermions. However, if we restrict our system to being two-dimensional, it was conjectured and then observed that certain "quasi-particles" can add an arbitrary complex phase $e^{i\theta}$ when exchanged. Particles whose corresponding change of phase is $e^{i\theta}$ for $\theta \neq \pm 1$ are known as **anyons**. Note that the term anyon comes from the exchange of such particles exhibiting *any* complex phase [1]. In this paper, I give a summary of the theoretical and experimental research in topological quantum computation, with an emphasis on the theoretical notions (as my main area of study is mathematics).

§2 Theoretical Efforts

The first seminal paper showing how to use anyons to simulate a quantum circuit was published by Kitaev in 1997 [2], and subsequently expanded by Ogburn and Preskill [3] in 1998. The following will be an attempt to summarize their results (first [2] and then [3]), using the original papers and lecture notes from Preskill [1].

§2.1 First Efforts

Both authors were interested in the Aharonov-Bohm effect. We see when an inaccessible flux "island" is located in a two-dimensional plane with charged particles surrounding the tube. As we move a charge around the island, its wave function will behave anyonicly! Furthermore, the acquired phase will be independent of the path taken up to homotopy, and is therefore referred to as a **topological** phase. To realize this effect using particles that have mass, we must use a material with extremely low resistance and with large amounts of flux. Using massive particles is important to maintain topological invariance due to the easily excitable nature of massless particles.

In the Aharonov-Bohm effect in two dimensions, we define the **topological spin** of a particle to be the phase factor its wavefunction gains from a complete counterclockwise rotation around the flux island. When we exchange two indistinguishable particles in this setup in a counterclockwise manner, the resulting change of phase from each is half of the phase change that each would acquire when rotation entirely around the island, and thus the wavefunction is multiplied by exactly the topological spin. We can furthermore that if an anyon has topological spin $e^{i\theta}$, its antiparticle in the same configuration would have topological spin $e^{-i\theta}$. If we have two groups of n indistinguishable anyons, the counterclockwise exchange of the two units will result in a phase shift of $e^{in^2\theta}$ for each of the two systems.

We can interpret these exchanges of particles as ribbons in spacetime as such: take the configurations of each of your particles in their state at time 0 and visualize them as small line segments in the plane, where the orientation of the segment corresponds to the orientation of the particles. As we progress in time, imagine raising the plane to height t at time t, moving each particles to its position and orientation at the specified time. Assuming we keep our line segments of small enough width so as not to intersect with other particles (which we can do since we endeavor to keep our particles sufficiently far apart to avoid accidental braiding), we can envision the resulting diagram as a braid of ribbons representing the particles' trajectories and orientations over time. The resulting strand from each particle is known as the particle's **worldline** and this process is known as **braiding**.

We will now give a brief overview about how Kitaev used these braiding anyons to derive a workable model of quantum computation in [2] and then go in more depth when discussing the work of Ogburn and Preskill. We can think of the phases obtained from braiding the worldlines of n anyons in fact as a representation of the braid group on n objects in the plane. More specifically, the **braid group** on n objects is defined as the possible nonintersecting braids we can form from the objects as they progress through time such that the braids do not intersect. The group operation is concatenation (i.e. perform one braid through time and then perform the next). This is because each braid corresponds to exactly one phase resulting from the braiding procedure. For background information on group representations, consult for instance [4] or [5]. If the resulting representation from exchanging two indistinguishable particles is one-dimensional, we call the anyons **abelian**. This terminology is as such because the group of 1×1 matrices is always abelian and in fact any representation of abelian groups can be decomposed into one-dimensional irreducible representations [4]. We call anyons **nonabelian** if the representation resulting from exchanging two indistinguishable anyons has dimension greater than 1, and is therefore a representation via a nonabelian linear group.

To further understand the representations of the braid group, we must first understand the group itself. The braid group, B_n, on n elements has the following presentation:

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} | \sigma_j \sigma_k = \sigma_k \sigma_j \text{ if } |j-k| \geq 2; \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \text{ if } j \neq n-2 \rangle$$

where σ_j is the counterclockwise swapping of item j with item $j+1$ [6]. In particular, the group is infinite and nonabelian. If our representation of anyons is abelian, we can immediately deduce from our presentation that

$$e^{i\theta_j} e^{i\theta_{j+1}} e^{i\theta_j} = e^{i\theta_{j+1}} e^{i\theta_j} e^{i\theta_{j+1}} \implies e^{i\theta} = e^{i\theta_{j+1}} \implies \theta = \theta_{j=1} \bmod 2\pi$$

giving that all indistinguishable particles must have the same topological spin, without us requiring it. However, this means that abelian anyons will not be particularly useful for quantum computation. In order to be able to approximate any unitary transformation on our vector space V_n our representation on n anyons acts on, we in fact need the operators of our representation to be dense in $SU(V_n)$ with respect to the operator norm (for more information, see for instance [7]). We will later see that the biggest restriction in finding such representations is that the fusion of three anyons must be associative, which will have corresponding effects on which phase shifts are allowed.

Returning to our use of the Aharonov-Bohm effect to find anyons, the isolated areas of magnetic flux in our superconductor correspond to some nonabelian group G. Furthermore, the charges we introduce act as unitary irreducible representations of G. In the notation of [1], moving a charge R one full loop around flux $a \in G$ corresponds to the operator

$$|R, j\rangle \mapsto \sum_{i=1}^{|R|} |R, i\rangle \, D_{ij}^{R}(a)$$

where $|R|$ denotes the dimension of the space R acts on and j is an element of our basis. It is here where we can actually see where our work corresponds to reality: we can actually measure $D_{ij}^{R}(a)$ by conducting interference experiments [2]. In fact, these representations of the group that we can measure actually allow us to associate specific elements of flux, or **fluxons**, with elements of the group up to conjugacy classes. This is because in fact irreducible representations the decomposition of representations of a group to irreducible ones depends only on the conjugacy classes of the group, as exhibited by the Peter-Weyl Theorem [8]. In fact, the indistinguishable fluxons are exactly those who are in the same conjugacy class in the resulting group. This is how [2], eventually arrives at a system in which we can actually initialize some known state and measure later on.

To understand how to actually perform quantum computation with the model we have arrived at, it is important to examine not just the effect of movement of one charge around a fluxon as we have done, but in fact how the system behaves when we exchange fluxons. To do this, we see how it affects paths of charged particles around these fluxons. If move a charge in path α around fluxon g_1 and path β around fluxon g_2, then assuming g_1 is "left" of g_2 as we follow counterclockwise around our two paths, we can see that interchanging g_1 and g_2 counterclockwise is in fact the same as mapping path $\alpha\beta\alpha^{-1} \mapsto \alpha$ and $\alpha \mapsto \beta$ [1]. We can furthermore deduce that this means the braid operator R must obey

$$R\,|a, b\rangle = |aba^{-1}, a\rangle$$

Because the product ab is preserved by R and R^{-1}, we can see that this action preserves the total flux, which in essence is what allows topological invariance of the system.

Through this lens of representations of groups, we can see that if every element has an order that divides n for some integer n, then the braiding operator R must have eigenvalues that are nth roots of unity in order to obey the definition of a representation. Furthermore, this braiding operator R allows for measurement since we can see from the fact that $R\,|a, b\rangle = |aba^{-1}, a\rangle$ that the if we braid a test charge b with $|a, a^{-1}\rangle$, then the result is a shifted superposition of all elements of a's conjugacy class, as laid out in [1]. Finally, as stated earlier, charges interact with our fluxons as unitary irreducible representations of the group, and we can represent each charge as an element of the tensor product decomposition of R and $\bar{R}$. If we combine a charged particle and flux, the conglomerate behaves an irreducible representation iff the charge and flux commute, and thus to consider them as an irreducible representation, we need to restrict to the normalizer of the charge in the group [1].

Now that we have background on the system, we can discuss how quantum computation is actually carried out, which is essentially just building up the parts we have already discussed. We will discuss in the next section a more general model for anyons and their interactions and how to compute with it, so we omit some of the details here. To create a known initial state, we introduce our fluxons and isolated charges, and introduce test charges to discover the conjugacy class of each of the fluxons. To simulate the application of gates, we apply our braiding operators (see the next sections for details), and then to measure the end result we can just fuse the resulting particles with test charges and see what residual charge or flux is left to see what particles we formed [1].

§2.2 Mathematical Generalizations

Subsequent to Kitaev's original paper, the anyonic model was generalized by [3] in a concise mathematical way and they demonstrated how to perform quantum computation without intermediate measurement. We lay out their mathematical generalization here and then their method of computation in depth in order to demonstrate the viability of topological quantum computers as a whole. For consistency, most of these definitions and arguments are taken from [1], but are similar or identical to those used in [3] originally (they are by the same author).

We define an **anyonic model** (also called a **unitary topological modular functor** in the more general categorical setting) M as consisting of an alphabet of labels $\mathcal{A}$ of allowed anyon types and their allowed charges, rules for fusing anyons $\times : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathbb{N}$ and the splitting of ions into fused components, and rules for exchanging, or vocabbraiding, anyons through specified unitary operators (ADD LATER). We further require a conjugation operation $\mathcal{A} \to \mathcal{A}$ denoted $a \mapsto \bar{a}$ taking a particle to its antiparticle. We denote the "empty particle" corresponding to no particle at all as 1 and require that $\bar{1} = 1$. Note that physically our labels and their operation are invariant under local changes to the physical system.

We write the fusion rule $\times$ as

$$a \times b = \sum_{c \in \mathcal{A}} N_{ab}^c c$$

where for every pair a, b we obtain a sum over all $c \in \mathcal{A}$ of nonnegative integers, and we require that $a \times b = b \times a$. As we will discuss later, we furthermore require fusion to obey an associative law. If $N_{ab}^c = 0$, we interpret that c cannot be obtained from the fusion of a and b, and similarly $N_{ab}^c = 1$ corresponds to c being obtained uniquely from a and b, whereas $N_{ab}^c > 1$ means that c can be obtained in distinguishable ways from a and b. The **fusion space** of a triple is the Hilbert space V_{ab}^c with the basis $\{|ab; c, \mu\rangle : \mu = 1, 2, ..., N_{ab}^c\}$.

An anyonic is **nonabelian** if there are nonunique ways to fuse anyons, i.e. if for some $a, b \in \mathcal{A}$, we have that

$$\dim(\bigoplus_c V_{ab}^c) \geq 2$$

We call $\bigoplus_c V_{ab}^c$ the **topological Hilbert space** of a and b.

Counterclockwise *braiding* of two anyons a, b is represented as a unitary isomorphism $R : V_{ba}^c \to V_{ab}^c$ between their fusion spaces, which we call the **braid operator**. Therefore, the operator R^2, called the **monodromy operator** represents the effect of winding a counterclockwise around b. Corresponding to our previous discussion, the **topological spin** θ_a of a label is the rotation obtained by taking the eigenvalue $e^{-i\theta_a}$ of $R_{a\bar{a}}^1$, and to be consistent with our earlier theory, we need that the eigenvalues of the monodromy operator $(R_{ab}^c)^2$ are of the form $e^{i(\theta_c - \theta_a - \theta_b)}$.

To describe the fusion of more than two anyons, we need to introduce what we call F-matrices. Since the total charge of a system of particles is observed to only depend on the particles themselves, we want fusion to be associative:

$$(a \times b) \times c = a \times (b \times c)$$

In particular, we can umbagiously write $V^d_{abc} = \bigoplus_e V^e_{ab} \otimes V^d_{eb} = \bigoplus_{e'} V^d_{ae'} \otimes V^{e'}_{bc}$ as a space, but we have two canonical bases: the one resulting from $V^d_{(ab)c}$ and the one resulting from $V^d_{a(bc)}$. We define the **F-matrices** (or **fusion matrices**) as the unitary map between these two bases.

Similarly, we can decompose an n-anyon system as

$$V^c_{a_1 a_2 \cdots a_n} = \bigoplus_{b_1,\ldots,b_{n-2}} (V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes \ldots \otimes V^c_{b_{n-2} a_n})$$

which we refer to as the **standard basis** and relate it to other bases by the F-matrices. This decomposition corresponds to considering first the fusion of a_1 and a_2, then the result of this fusion fused with a_3, etc.

To describe braiding of more than just two anyons, and therefore allow us to perform what will become multi-qubit gates, we need to introduce what is know as the B-matrix of the anyons. The problem with the R-matrix is that we are only taking into account the effect on just the two anyons we are swapping, which might not be the only effect on the entire system. To remedy this, we can make the key observation that the R matrix of two anyons is block-diagonal when we change bases with the F-matrix. Therefore, we just define the **B-matrix** of the braiding of two anyons to be the composition of their F-matrix, their R-matrix, and then the inverse of their F-matrix. More concretely, we define the B-matrix with the following terribly messy notation:

$$(B^d_{abc})^g_e = \sum_f [(F^{-1})^d_{abc}]^g_f R^f_{bc} (F^d_{abc})^f_e$$

Equipped with these matrices, we have entirely specified the action of the braid group on our fusion spaces and we are prepared to talk about specific examples and how we can actually perform computations with them.

§2.3 Computing with Anyons: The Fibonacci Model

We now have the machinery to describe the specific anyonic model that Obgurn and Preskill found in [3]. We will continue using the notation of [1] for more clarity. The particular nonabelian anyonic model is the **Fibonacci** (or **Yang-Lee**) model. The behavior of this model has been observed in a physical system [SOURCE], as we will discuss in later sections. In this model, we define $\mathcal{A} = \{0, 1\}$ (where now the identity is denoted as 0), and our only fusion rule is

$$1 \times 1 = 0 + 1$$

Note that $\dim(V^0_{11} \oplus V^1_{11}) = 2$, making our model nonabelian.

First we determine what the fusion spaces of this model are. Consider $V^b_{1^n}$ for a b in our alphabet $\{0, 1\}$. Fusion spaces of this form are the only nontrivial ones due to our fusion rule. Looking at our standard basis decomposition, we can completely describe the sequence of fusions by the intermediate charges $b_1, \ldots, b_{n-2}$. If we consider just $V^0_{1^n}$, we can see from our fusion rules that in order to get a final charge of 0 we need $b_{n-2} = 1$ because the final anyon fused is a 1, which only can fuse to 0 if the result of the subsequent fusions is 1. Furthermore, we can by the same argument recursively we can see that we can never have two 0s in a row in our sequence $b_1, \ldots, b_{n-3}$ of fusion outcomes because otherwise we would not be able to have a charge of 0 at the end. However, these are the only restrictions we need, so our $b_1, \ldots, b_{n-3}$ are in bijection with the set of strings with no consecutive zeros. The dimension of this space therefore satisfies:

$$N^0_1 = 0$$

$$N^0_{1^n} = N^0_{1^{n-1}} + N^0_{1^{n-2}}$$

corresponding to the cases where the first term is 1 and 0 respectively. This recurrence is the reason this model is known as the *Fibonacci* model.

We now describe the R and F-matrices for this model. We can in fact deduce what the R matrices must be from the F matrices and the canonical isomorphisms we must have.

Finally, we discuss how to use the Fibonacci model for quantum computation. First note that from our Fibonacci recurrence, we can see that $\dim(V^0_{1^4}) = 2$, and can therefore be used as a single qubit.

To show how we can actually access the braiding properties of the system, we need to introduce fundamental relations known as the **pentagon equation** and **hexagon equation**. These will be referenced in later sections and in more generality, so it is important to discuss their formulations here. Included in the appendix is diagrams corresponding to the pentagon and hexagon equations, giving the reader an idea of why they are named as such.

The pentagon and hexagon equations simply state that the five distinct ways we are able to fuse four anyons (i.e. 2 qubits), and the six ways we could fuse six anyons must all have a canonical isomorphism between them. In its entirety, the pentagon equation says that given anyons $1, 2, 3, 4$ their F-matrices must satisfy the following relation:

$$(F^5_{12c})^d_a (F^5_{a34})^c_b = \sum_e (F^d_{234})^c_e (F^5_{1e4})^d_b (F^b_{123})^e_a$$

Similarly, we can derive a consistency equation that the braiding of six anyons must satisfy, the hexagon equation:

$$R^c_{13} (F^4_{213})^c_a R^a_{12} = \sum_b (F^4_{231})^c_b R^4_{1b} (F^4_{123})^b_a$$

In fact, it is shown in [10] that the pentagon and hexagon equations are the only relations we need to guarantee consistency of the braiding operator with the fusion operator. This result is a special case of the MacLane Coherence Theorem for fusion categories [10].

Armed with the pentagon and hexagon equations, we are prepared to deduce what the R-matrices and F-matrices for our Fibonacci model must actually be. Solving the pentagon equation, we can see that the two F matrices on our fusion spaces are:

$$(F_{0111})^b_a = \delta^1_a \delta^b_1$$

$$F_{1111} = \begin{pmatrix} \tau & e^{i\phi}\sqrt{\tau} \\ e^{-i\phi}\sqrt{\tau} & -\tau \end{pmatrix}$$

where τ is one less than the golden ratio and ϕ is an arbitrary phase (see [1] for details). Using the hexagon equation, we furthermore get that the R-matrix of exchanging two adjacent anyons is:

$$R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}$$

with F-matrix

$$F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

(again, see [1] for details; we include these results here in order to be able to discuss actual computation with Fibonacci anyons).

At last, we are ready to derive a model for quantum computation from our anyonic model. As noted earlier, we will encode a qubit as four anyons, since the dimension of the fusion space of four anyons is the fourth Fibonacci number: $\dim(V^0_{1111}) = 2$. We can realize the basis state $|0\rangle$ as the case where $V^0_{11} \otimes V^0_{011}$ where the first two anyons fuse to form a 0 anyon, and similarly we denote $|1\rangle$ as the case where the result of this fusion is instead 1: $V^1_{11} \otimes V^0_{111}$. This is valid because, as discussed, $V^0_{1111} = V^0_{11} \otimes V^0_{011} \oplus V^1_{11} \otimes V^0_{111}$. Now, the exchange of the first two qubits, which [1] denotes as σ_1, acts on our one-qubit space as

$$\sigma_1 = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}$$

(derived from the pentagon equation), and furthermore the B-matrix is

$$\sigma_2 = F^{-1}RF = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & \tau \end{pmatrix}^{-1} \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix} \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & \tau \end{pmatrix}$$

and in fact these matrices together are dense in $SU(2)$, so they can be used to approximate any single qubit operation! In fact, our braiding operation acting on two sets of four anyons (i.e. a two-qubit system) has a B-matrix acting in the same manner, so we get universal computation for free. The details are carried out in [3], and involve more powerful machinery than introduced here.

§2.4 Ising Model

The other anyon model that is believed to be most viable for physical realization is the **Ising** model. We will discuss its properties here, with [9] as our reference. The Ising model consists of the alphabet $\mathcal{A} = \{1, \psi, \sigma\}$ of particles. The nontrivial fusion rules are:

$$\psi \times \psi = 1; \psi \times \sigma = \sigma; \sigma \times \sigma = 1 + \psi$$

We can see that $\dim(V_{\sigma\sigma}) = 2$, meaning our model is nonabelian. We can see that the fusion of three σ particles must result in a σ charge:

$$(\sigma \times \sigma) \times \sigma = (1 + \psi) \times \sigma = \sigma + \sigma$$

To compute the F-matrix of the nontrivial fusion spaces, we consider the effect of fusing the leftmost σ particles first in a triple or the rightmost:

$$(\sigma \times \sigma) \times \sigma : \{|1 \times \sigma; \sigma\rangle, |\phi \times \sigma; \sigma\rangle\}$$

$$\sigma \times (\sigma \times \sigma) : \{|\sigma \times 1; \sigma\rangle, |\sigma \times \phi; \sigma\rangle\}$$

and so in particular the two bases we have correspond to

$$V^\sigma_{(\sigma\sigma)\sigma} = (V^1_{\sigma\sigma} \otimes V^\sigma_{1\sigma}) \oplus (V^\psi_{\sigma\sigma} \otimes V^\sigma_{\psi\sigma})$$

$$V^\sigma_{\sigma(\sigma\sigma)} = (V^\sigma_{\sigma 1} \otimes V^1_{\sigma\sigma}) \oplus (V^\sigma_{\sigma\psi} \otimes V^\psi_{\sigma\sigma})$$

and therefore the F-matrix between these canonical bases is $F : V^\sigma_{\sigma\sigma\sigma} \to V^\sigma_{\sigma\sigma\sigma}$ defined by

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Note that since this matrix is self-invertible it is unnecessary to which basis we are starting or ending with. Additionally, we can see that the difference between computing in the two different bases is equivalent to computing in the standard basis vs. computing in the Hadamard basis in our familiar model of quantum computing.

We can similarly computing the braiding operators of exchanging two σ particles. By definition, it is the direct sum of the two maps $R^1 : V^1_{\sigma\sigma} \to V^1_{\sigma\sigma}$ and $R^\phi : V^\phi_{\sigma\sigma} \to V^\phi_{\sigma\sigma}$, which map $1 \mapsto e^{-i\frac{\pi}{8}}$ and $1 \mapsto e^{-i\frac{3\pi}{8}}$, which we can find by again solving the hexagon equations. In particular, we have that

$$R = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{-i\frac{3\pi}{8}} \end{pmatrix}$$

One of the fundamental shortcomings of the Ising model in comparison to the Fibonacci model is that we can *not* implement universal gates just with braiding operations. Using the R-matrix, we can only implement a relative phase change, (it is the same for the R-matrix of Fibonacci anyons). The F-matrix allows the X gate

$$F^{-1}R^2F = e^{-4\pi i}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

and in fact, it is shown in [9] that the gates generated by these gates (the Clifford group) are the *only* ones we can realize with the Ising model. In particular, we cannot perform universal quantum computation in this model.

However, the Ising model does provide one attractive feature the Fibonacci model does not: it admits a natural tensor product decomposition. This is because as we add another σ anyon, the dimension always doubles, and so we can view the entire space as a tensor product of the fusion space of the $n-1$ σ anyons and the fusion space of the last σ. Computation with the Ising model is similar to computation in the Fibonacci model except with the obvious limitations.

§3 Experimental Realizations

Up until now, we have mainly restricted ourselves to theoretical ways in which fault-tolerant quantum computation computation could be carried out using anyons without discussing experimental observations of such anyons. Presently, the only circumstances under which behavior similar to nonabelian anyons has been observed is in quasiparticles formed through the fractional quantum hall effect. In this section we give a brief overview of the effect, and how it has been observed to produce desirable anyons.

§3.1 Fractional Quantum Hall Effect

According to [14], currently the only known physical observations of nonabelian anyonic behavior is via the fractional quantum Hall effect in superconductors. Since I assume little physics background on the part of the reader (as I have), I now give a quick description of my understanding of the fractional quantum Hall effect and its properties due to [14]. The classical Hall effect is when a magnetic field perpendicular to current flowing in some conductor causes a current in the transverse direction. This makes sense because we recall from basic physics that moving charges in magnetic fields experience forces from the field in the direction respecting the "right-hand" rule.

In the quantum mechanical setting, the conductance resulting from this effect will take on specific quantized values obeying the equation

$$\sigma = \nu\frac{e^2}{h}$$

where σ is the conductance, e is the charge of an electron, h is Planck's constant, and ν is a constant factor (a rational number) depending on the situation, known as the **filling**

factor. If ν is not an integer, we call the situation the **fractional** quantum Hall effect. The charges we see occurring are actually electron-electron interactions and not full particles like we would normally expect. They appear like small "whirlpools" of current in the material that we refer to as **vortices**. Since the particles are not full particles in the usual sense, we refer to them as **quasiparticles**.

The quantum Hall effect is generally observed in systems which are constrained to remain in two-dimensions, and in low temperatures and strong magnetic fields. There, we often refer to the electrons as behaving as a "gas" with vortex excitations. Therefore, it is in situations like this where we will see that anyonic behavior has been observed.

§3.2 Anyons Observed

We now give an overview of how the fractional quantum Hall effect has actually been used to observe behavior resembling nonabelian anyons and give references to the major advances in this area.

The first wavefunction that was believed to behave as a nonabelian anyon was discovered by Moore and Read in 1990 [15]. The wavefunction came to be known as the **Moore-Read** wavefunction and they referred to the particles in such a system as "nonabelions." Futhermore, they believed that a previously-known example of the fractional quantum Hall effect in fact had the Moore-Read wavefunction. This was the instance found by Willett, et. al. in 1987 that had filling factor $\nu = \frac{5}{2}$, which was a notable result by itself, being the first known instance of the fractional quantum Hall effect where the filling factor had even denominator.

In the 2000s, there were several further efforts to verify that the Willett example with $\nu = \frac{5}{2}$ is nonabelian. One such experiment was carried out in 2005 by Kitaev and several collaborators in [16]. Another effort was that of Stern and Halperin in [17]. Both authors proposed using an interferometer to detect interference patterns and see if the resulting distributions were nonablian. In particular, [16] showed that in their experiment the period of oscillations in the resistance of the conductor they were measuring would depend on where the number of quasiparticles is odd or even exactly in the case that the quasiparticles are nonabelian.

More recently, Camino, et. al. have found further evidence of the Aharonov-Bohm effect (discussed earlier) occurring in fractional quantum Hall states with filling factor $\nu = \frac{5}{2}$ in a way that supports the nonabelian model of their interactions [18]. In [19], Bonesteel, et. al. showed in practice that the $\nu = \frac{5}{2}$ can actually be used to implement a NOT-gate and that the resulting system has an error rate of 1 in 10^6. Further verification of the above experiments has since occurred and it is still believed that the $\nu = \frac{5}{2}$ fluid behaves as non-abelian anyons. In fact, the behavior that has been observed up to the present indicates that this fractional quantum Hall state corresponds to the Ising anyonic model that we have previously developed [10].

The other (and newest) main area of investigation is a particular fractional quantum Hall state with filling factor $\nu = \frac{12}{5}$. In recent years, works such as [20] have demonstrated that to the best of our knowledge this system behaves like our Fibonacci anyonic model. This is clearly of much greater interest than the $\nu = \frac{5}{2}$ state due to its ability to perform universal quantum computation. However, this $\nu = \frac{12}{5}$ state has proved to be much more unstable and harder to create than the $\nu = \frac{5}{2}$ states that were previously studied [10].

§4 Further Theory

For a the most in-depth (and relatively recent) treatment of the current theoretical state of the field of topological quantum computing, consult [10].

§4.1 Toric Codes

One application for abelian anyons is the so-called toric stabilizer codes for storing quantum information. We try to summarize them here and note that they are discussed at length in [2], [10], and [13].

The basic idea behind the construction is that we place a square lattice on a torus and on each edge we place our elements of flux. In [13], it is shown that if we are able to do so with a fractional quantum Hall state with filling factor $\nu = \frac{1}{3}$, then the resulting system will store a qutrit of information and be almost completely immune to local pertubations. Note that the anyons in this case are abelian, showing that though abelian anyons are of no use for actual quantum computation, they can be used to safely store quantum information. An extensive discussion of the toric codes and their implementation are ommitted for brevity.

§4.2 Categorical Formulations

In [10], Wang related the constructions previously made in connection with topological quantum computing with topological quantum field theory, and furthermore to categorical constructions (often referred to as "abstract nonsense"). We give a brief tour of the definitions made and the ability to derive useful results from them.

A **fusion category** $\mathcal{C}$ is defined as a monoidal (has a tensor product) category that is additionally $\mathbb{C}$-linear, rigid (has a natural dual structure), semisimple (has simple objects of which each object is a direct sum), and has finitely-many isomorphism classes of simple objects which include the unit. A **ribbon fusion category** is a category that admits a ribbon structure satisfying the hexagon rule as described earlier. A fusion category is **unitary** if there is a conjugation action such that if the composition of an object and its conjugate is the identity, then the object itself is the identity. A ribbon fusion category is a **modular tensor category** if the determinant of the quantum dimension matrix (definition omitted for brevity) is nonzero [10]. As we can see, these definitions all agree with our prior definitions of a fusion space and its associated operations and properties.

In [11], it was shown that there are 35 unitary modular tensor categories of rank ≤ 4, and that modular tensor categories can be characterized by their "Witt classes", and in particular a modular tensor category is equivalent to the trivial one iff it is the center of some trace category. In 2013, [12] showed that in fact that the braiding structure of a unitary fusion category must be unitary and that each unitary braided fusion category has exactly one unique unitary ribbon structure.

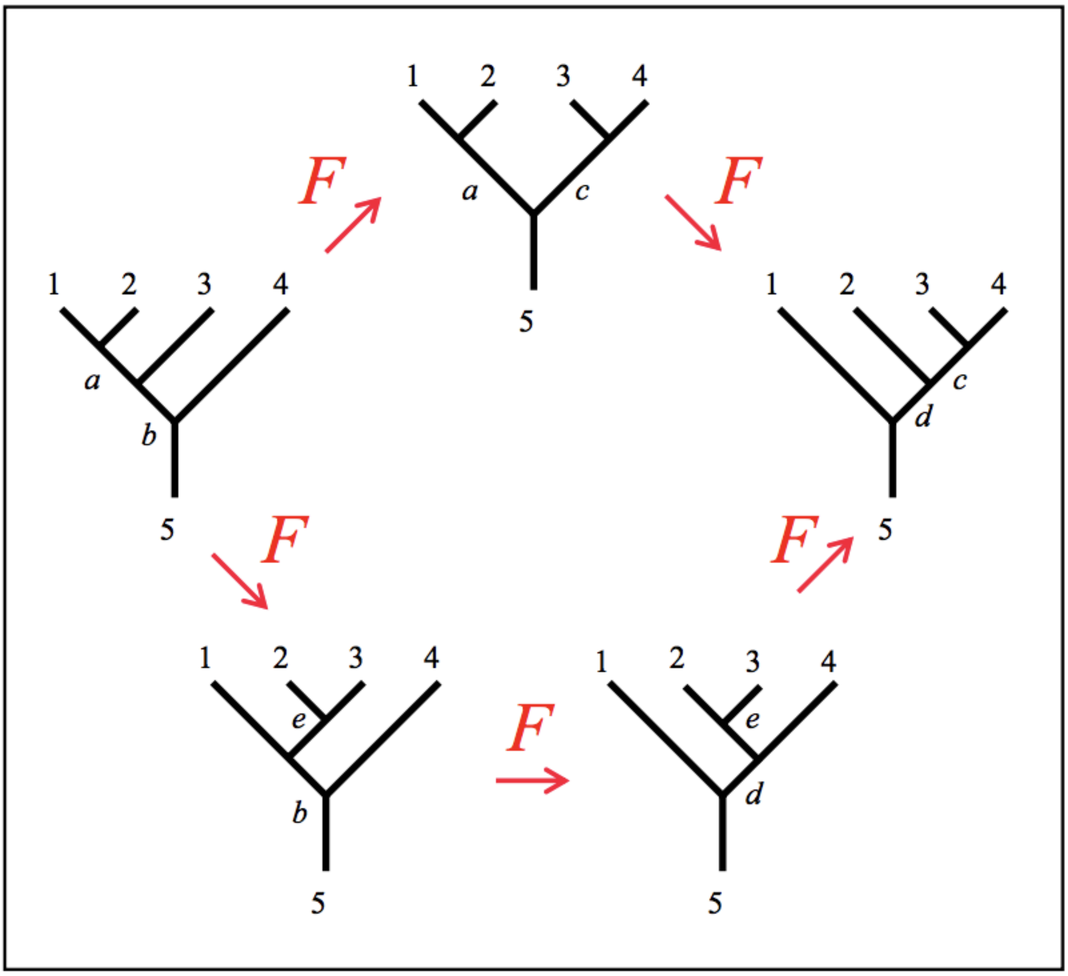
According to [10], continuing open problems in this area include further classification of low-rank modular tensor categories, the classification of modular tensor categories up to Morita equivalence (which is essentially when the category of modules, a generalization of vector spaces, over the categories are equivalent; for more information see for instance [5]), and a categorical formulation of topological phase transitions as quotients of tensor functors.

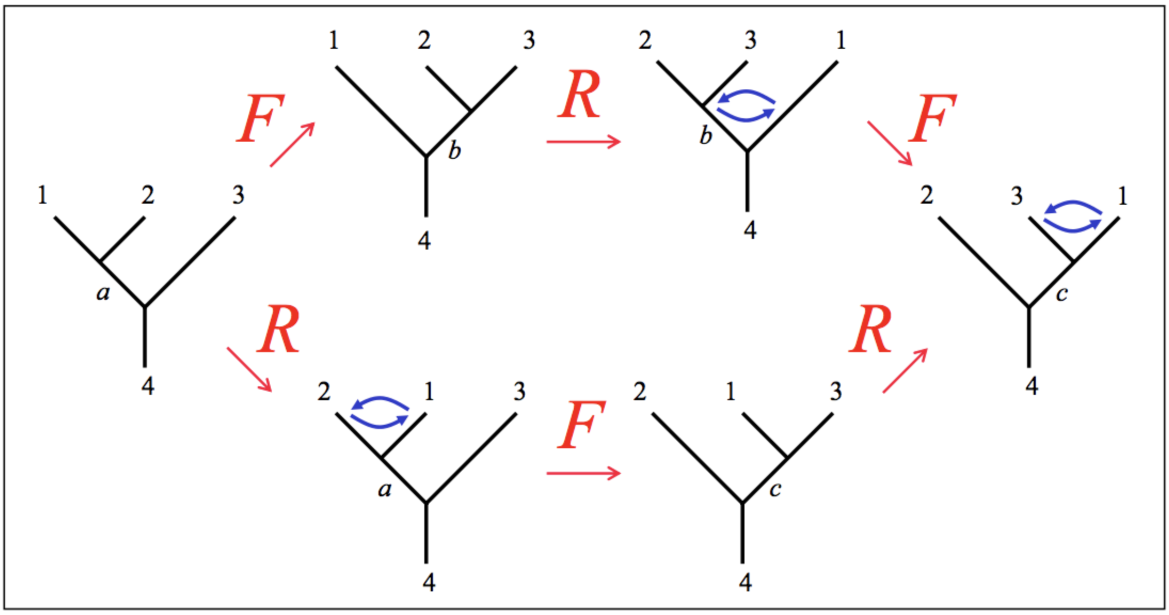
§5 Conclusion

In this paper, we first described anyons as particles whose wavefunction can pick up nontrivial phases upon exchange and then discussed how braiding them could lead us to quantum computation. We did this by first formalizing a model of an anyonic system and then showing specific examples of a Fibonacci model which can perform universal computation. We additionally showed that a more simple Ising model could be used to at least store quantum information. Furthermore, we discussed experimental attempts to realize these theoretical advances in superconductors using the fractional quantum Hall effect. Lastly, we demonstrated further advances that are being made in the theory of these topological systems and several open problems. Topological quantum computation, if feasibly realizable, would provide a simple solution to the decoherence problems of more traditional approaches to quantum computation.

§6 Appendix: Diagrams

The following diagrams represent the different ways in which systems of four anyons or six anyons could fuse. They must be canonically isomorphic to correspond with reality. They are known, respectively, as the pentagon and hexagon equations. Both diagrams are taken from [1].





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