Kazhdan-Lusztig Theory

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1 Verma Module Characters

Recall last week we introduced the Category $\mathcal{O}$ for a Lie algebra and the Verma module simple quotients $M(\lambda) \to L(\lambda)$, $\lambda \in \mathfrak{h}^*$ a weight, were exactly the simple objects of $\mathcal{O}$. When these quotients were finite-dimensional (i.e. λ dominant integral) we had the Weyl Character Formula

Theorem 1 (Weyl Character). *For $L(\lambda)$ finite-dimensional,*

$$Ch(L(\lambda)) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot 0}} \quad \textit{(Note } w \cdot \rho = w(\lambda + \rho) - \rho\textit{)}$$

for W the Weyl group, ρ the half-sum of positive roots, and formal character

$$Ch(M) = \sum_{\lambda \in \mathfrak{h}^*} dim(M_\lambda) e^\lambda$$

We wanted to compute the composition factors of other Verma modules and we had obtained that using translation functors we can reduce the computations to $\mathcal{O}_0$. Specifically in the Grothendieck group $K_0(0)$

$$[L_w] = \sum_{y \leq w} a_{x,w} [M_y]$$

where $M_w = M(w \cdot (-2\rho)), L_y = L(y \cdot (-2\rho))$ (ρ is the half sum of positive roots) and these a_x, w are integers. The surprising fact is that these $a_{x,w}$ are some $P_{x,w}(1)$ for $P_{x,w} \in \mathbb{Z}[q]$ that come completely from combinatorics of the flag variety.

Example 1. Let's recall what happens for $\mathfrak{sl}_2 = \langle e, f, h \rangle$ with $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. The finite-dimensional irreducibles are all found in the representation on functions (where e, f, h act concretely by $e = x\frac{\partial}{\partial y}, f = y\frac{\partial}{\partial x}, h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$)

$$\text{Fun}(\mathbb{C}^2) = \bigoplus_{n \geq 0} L_n \boxtimes \mathbb{C}(x, y), \quad L_n = \text{span}\{x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n\}$$

The computation for the character of L_n directly is just (the weight spaces are 1 dimensional and go from n to $-n$ stepping by two):

$$\mathrm{Ch}(L_n) = e^n + e^{n-2} + \cdots + e^{-n+2} + e^{-n} = \frac{e^n}{1-e^{-2}} - \frac{e^{-n-2}}{1-e^{-2}} =$$

$$= \mathrm{Ch}(M_n) - \mathrm{Ch}(M_{-n-2}) = \mathrm{Ch}(M(\lambda_n)) - \mathrm{Ch}(M(s\cdot\lambda_n))$$

which is the Weyl character formula form.

Notice that these $L_n = \Gamma(\mathbb{P}^1, \mathcal{O}(n))$, which will be relevant when thinking about the connection between differential operators on $\mathbb{P}^1$ and $\mathfrak{sl}_2$.

2 Combinatorial Picture

A Coxeter group is a group generated by reflections (order 2 elements) $\{s_1, \ldots, s_n\}$ with relations of the form $(s_i s_j)^{m_{i,j}} = 1$. Weyl groups are a particular example. Remember that you can get the Weyl group from G as $N_G(T)/T)$ for T a maximal torus in a Borel subgroup. For GL_n or $\mathfrak{sl}_n$ the Weyl group is just the symmetric group on n letters.

Let W be the Weyl group (or any Coxeter group) and S the set of generating reflections. For S_n this is the set $\{(12), (23), \ldots, (n-1\ n)\}$ of simple transpositions. The length of a Weyl group element is the fewest number of simple reflections in an expression for the element and elements are ordered $y \leq w$ means $w = ys_1 \cdots s_k$ with each right multiplication increasing the length. Explicitly the length in S_n can be counted as the quantity of pairs in $\{1, \ldots, n\}$ that the permutation inverts the order of ($i < j$ becomes $j < i$).

Definition 1. The **Hecke algebra $\mathcal{H}_W$ of W** is the free $\mathbb{Z}[q^{\pm 1/2}]$-module (we'll see why we need square roots) on $\{T_w\}_{w\in W}$ with identity T_1 and multiplication via

$$\begin{cases} T_s T_w = T_{sw} \text{ for } \ell(sw) > \ell(w) \\ T_s T_w = (q-1)T_w + qT_{sw} \text{ for } \ell(sw) < \ell(w) \end{cases} \tag{1}$$

Notice that if we set $q = 1$ both relations collapse to

$$T_s T_w = T_{sw}$$

, so the Hecke algebra is a deformation of the integral group algebra $\mathbb{Z}[W]$ of W.

However, in the groups algebra T_s squares to 1 (because $s^2 = 1$), but in the Hecke algebra

$$\boxed{T_s^2 = (q-1)T_s + qT_1}$$

We can still make T_s invertible because

$$T_s\,(aT_1 + bT_s) = aT_s + b\,((q-1)T_s + qT_1) = (a + b(q-1))T_s + bqT_1$$

$$\implies \text{ by setting } b = q^{-1}, a = q^{-1} - 1 \text{ we have } T_s\left((1-q^{-1})T_1 + q^{-1}T_s\right) = T_1.$$

Together with

$$((q^{-1}-1)T_1 + q^{-1}T_s)T_s = q^{-1}T_s - T_s + q^{-1}(q-1)T_s + q^{-1}qT_1 = T_1$$

we have that $\boxed{T_s^{-1} = (q^{-1}-1)T_1 + q^{-1}T_s}$.

Since every element w of the Weyl group is generated by a (nonunique) minimal string of generators in S, $w = s_1 \cdots s_n$, we have

$$T_w = T_{s_1 \cdots s_n} = T_{s_1} \cdots T_{s_n} \implies T_w^{-1} = T_{s_n}^{-1} \cdots T_{s_1}^{-1} \implies \boxed{\text{all } T_w \text{ are units}}$$

An inductive argument shows the formula for T_s^{-1} generalizes as

$$(T_{w^{-1}})^{-1} = (-1)^{\ell(w)} q^{-\ell(w)} \sum_{y \leq w} (-1)^{\ell(y)} R_{y,w}(q) T_y, \;\; R_{y,w} \in \mathbb{Z}[q] \text{ of degree } \ell(w) - \ell(y)$$

Definition 2. The **<u>bar involution</u>** on $\mathcal{H}_W$ is given by

$$\begin{cases} \bar{q} = q^{-1} \\ \overline{T_w} = (T_{w^{-1}})^{-1} \end{cases} \tag{2}$$

Can we find a basis of $\mathcal{H}_W$ indexed by W still and stable under the involution? For s a simple reflection

$$\overline{A(q)T_1 + B(q)T_s} = A(q^{-1})T_1 + B(q^{-1})((q^{-1}-1)T_1 + q^{-1}T_s) =$$

$$= \Big(A(q^{-1}) + B(q^{-1})(q^{-1}-1)\Big)T_1 + q^{-1}B(q^{-1})T_s$$

So for self-duality we want

$$B(q) = B(q^{-1})q^{-1} \quad \text{and} \quad A(q) - A(q^{-1}) = B(q^{-1})(q^{-1}-1) = q^{1/2}(q^{-1}-1) = (-q^{1/2}) - (-q^{-1/2})$$

$$\implies B(q) = q^{-1/2}, A(q) = -q^{1/2}$$

$$\implies \boxed{C_s = -q^{-1/2}T_1 + q^{-1/2}T_s}$$

Notice that we needed square roots of q to make this calculation work. The nicest coefficients for an invariant basis we can get will be of the following form:

Theorem 2 (due to Kazhdan-Lusztig)**.** *There is a **<u>unique</u>** basis $\{C_w\}_{w \in W}$ of $\mathcal{H}_W$ fixed under the involution subject to the normalization:*

1. $C_w = (-1)^{\ell(w)} q^{\ell(w)/2} \sum_{y \leq w} (-1)^{\ell(y)} q^{-\ell(y)} \overline{P_{y,w}(q)} T_y$ *for* $\boxed{P_{y,w} \in \mathbb{Z}[q]}$
2. $P_{w,w} = 1$ *and deg* $P_{y,w}(q) \leq \frac{\ell(w)-\ell(y)-1}{2}$

Example 2. • For S_2, $C_1 = T_1$ and

$$C_s = q^{-1/2}(T_s - qT_1) = -q^{1/2}(T_1 + (-1)q^{-1} \cdot \bar{1} \cdot T_s)$$

$$\implies P_{1,s} = 1$$

- For S_3, the group elements are $1, s_1 = (12), s_2 = (23), s_1s_2, s_2s_1, s_1s_2s_1$. The degree bound says the biggest degree we could have is $\frac{3-0-1}{2} = 1$ if y trivial and otherwise degree 0. But $P_{1,w} = 1$ because

$$\overline{T_1} = T_1 \implies (-1)^0 q^{-0} P_{1,w}(q^{-1}) = (-1)^0 q^0 P_{1,w}(q)$$

$$\implies P_{1,w}(q^{-1}) = P_{1,w}(q) \implies P_{1,w}(q) = \text{constant}$$

  So in fact all polynomials are 1 for S_3.

- For S_4, will get the first nontrivial polynomials which are each $P_{y,w}(q) = q + 1$ when nontrivial. One of the nontrivial ones is $P_{t,tsut}$ for $S_4 = \langle s, t, u\rangle$ which does have degree bounded by $\frac{4-1-1}{2} = 1$.

- **Theorem due to Polo [1999]**: Any polynomial of degree d with integer nonnegative coefficients is the Kazhdan-Lusztig polynomial of some pair y, w in in the symmetric group of order $1 + d + P(1)$. So K-L polynomials are arbitrarily bad.

Idea of Proof of Theorem. Induct on the length of w. If $w = w's$, then try just multiplying together the $C_{w'}, C_s$ and then do a series of corrections to get better expressions. □

The Kazhdan-Lusztig Conjecture is that if we evaluate back at $q = 1$ (the same value where the Hecke algebra became the group algebra) we get the coefficients in our multiplicity formula.

Theorem 3 (Kazhdan-Lusztig "Conjecture")**.** *In the principal block $\mathcal{O}_0$ and simple quotient Verma module $M_w \twoheadrightarrow L_w$ where $M_w = M(w \cdot (-2\rho)), L_w = L(w \cdot (-2\rho))$ (all of the simple modules in $\mathcal{O}_0$) for $w \in W$ Weyl group,*

$$[L_w] = \sum_{y \le w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)[M_y]$$

These Kazhdan-Lusztig polynomials originally came from what I'll talk about in the next section. The really surprising part was that they relate to Verma module multiplicities.

3 Geometry

Where do the Kazhdan-Lusztig polynomials come from originally? My aim is to show it relates to some cohomological data about subsets of the flag variety:

$$P_{y,w}(q) = \sum_{i=0}^{\ell(w)} \dim IH^{2i}_{ByB}(\overline{BwB})$$

They are supposed to tell you about the combinatorics of singularities in Schubert varieties (their intersection cohomology). Recall that our original setup was we take our group G, say $G = GL_2$, and a Borel subgroup B, say the subgroup of upper triangular matrices, T a maximal torus in G:

$$G = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

The Weyl group is then $N_G(T)/T$ (in this case permutation matrices). The flag variety is then G/B. For GL_2 this $\mathbb{P}^1$. Finally we had the Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB$$

The **Schubert cells** B_w are the image in the flag variety of BwB and their closures are the **Schubert varieties** X_w, each cell is isomorphic to $\mathbb{A}^{\ell(w)}$. The Weyl group ordering tells us geometrically about the inclusions, a stratification by affine opens:

$$X_w = \bigcup_{y \leq w} B_y$$

(and $B_y \subseteq X_w$ exactly when $y \leq w$). For GL_2 this is just the point and inclusion into $\mathbb{P}^1$. The first correspondence between the combinatorics above and geometry is

$$\boxed{T_w \sim j_!(\underline{\mathbb{C}}_{B_w})}$$

where $j_!$ is the shriek extension (extension by 0) of the constant sheaf.

Recall that in previous talks we got q terms from the action of the Frobenius over nonzero characteristic. This Frobenius map is a canonical endomorphism on varieties over nonzero characteristic that on $\mathbb{P}^1$ is given on closed points by $[x : y] \mapsto [x^p : y^p]$.

What does this do on cohomology? For $\mathbb{P}^1$ the nonzero cohomology degrees are (in standard cohomology) $H^0(\mathbb{P}^1) = \mathbb{C}$ and $H^2(\mathbb{P}^1) = \mathbb{C}$. The Frobenius acts trivially on 0th cohomology (since it's canonically defined and acts trivially on a point). On top cohomology,

$$[x : 1] \mapsto [x^p : 1] \implies \deg(Fr) = p \implies \text{acts by } p$$

Similarly using something like Kunneth we have that $H^*(\mathbb{P}^1 \times \mathbb{P}^1)$ is

$$H^0 = \mathbb{C} \quad H^1 = 0 \quad H^2 = \mathbb{C} \oplus \mathbb{C} \text{ acts by } p \quad H^3 = 0 \quad H^4 = \mathbb{C} \otimes \mathbb{C} \text{ acts by } p^2$$

Therefore, roughly we expect Frobenius acts on ith cohomology by $q^{i/2}$ (I'll now use q instead of p).

To make this actually work we need to instead use intersection cohomology because our Schubert varieties will have singularities. Concretely, we want to take ($R\Gamma(\cdot)$) sections of a sheaf called the IC (intersection cohomology) sheaf instead of the constant sheaf. In particular, in the category of B-constructible perverse sheaves on G/B, the IC_w are the simple objects. Perverse sheaves lived in a derived category so these aren't sheaves per se but complexes of sheaves.

Without giving an exact definition of what the IC sheaf is, I'll draw a picture of what it does. If we have some singularity that looks like two $\mathbb{P}^1$'s meeting at a point (like two spheres glued together) then the stalks away from that singularity point should agree with the constant sheaf, and therefore are $\underline{\mathbb{C}}$. But at the singularity we should have stalk $\mathbb{C} \oplus \mathbb{C}$. So the IC sheaf is roughly

$$IC_{\text{space}} = \underline{\mathbb{C}}_{\text{left sphere}} \oplus \underline{\mathbb{C}}_{\text{right sphere}}$$

This is where the next ingredient in the combinatorics comes from:

$$\boxed{C_w \sim IC_{X_w}}$$

This lines up with our understanding of S^2 because the flag variety is $\mathbb{P}^1$ and the IC sheaf will just be the constant sheaf ($\mathbb{P}^1$ has no singularities!), so the constant sheaf is the "sum" of the extension by 0 of the constant sheaf on the point $B1B = B$ (we're working in G/B so this becomes a point) and the constant sheaf on the other cell BsB:

$$\boxed{IC_s \sim \delta_e + \delta_s \iff C_s \sim T_e + T_s}$$

Where does the bar involution come from? Recall that for sufficiently nice manifolds Poincare duality gives an isomorphism between ith cohomology and $n-i$th cohomology for n the top degree. This generalizes to intersection cohomology for IC sheafs and the map we get is the bar involution

$$\boxed{\text{bar involution} \sim \text{Verdier duality}}$$

The precise statement is that you can form the character of a perverse sheaf $\mathcal{F}$ on the flag variety of roughly the form

$$\mathrm{Ch}(\mathcal{F}) = \sum_{w\in W, k\in\mathbb{Z}} (\dim H^{-\ell(w)-k}) q^k \delta_w$$

and under the bar involution for sufficiently nice $\mathcal{F}$,

$$\boxed{\mathrm{Ch}(\mathbb{D}\mathcal{F}) = \overline{\mathrm{Ch}\mathcal{F}}}$$

so our character formula is self-dual.

For a general IC sheaf, Frobenius will act on degree $2i$ by q^i, and so if we label the rank in each degree as $\underline{\mathbb{C}}^{a_{2i}}$, then when we take trace of the Frobenius operator we get a polynomial

$$\boxed{Tr(Fr) = 1 + a_1 q + a_2 q^2 + \cdots \sim \text{Kazhdan-Lusztig Polynomial}}$$

In particular the $P_{y,w}$ will come from the trace of the Frobenius on $i^*_{ByB} IC_{\overline{BwB}}$. This is how the Kazhdan-Lusztig polynomial is telling us about the combinatorics of singularities in the Schubert varieties.

Example 3. Recall that the first nontrivial K-L polynomial we get is in $S_4 = \langle s, t, u\rangle$ for K-L polynomial $P_{t,tsut}(q) = q + 1$. Geometrically this is because looking at

$$i^*_{BtB} IC_{tsut} \to \text{get constant sheaf } \underline{\mathbb{C}} \text{ away from the copy of } \mathbb{P}^1_t$$

$$\to \text{get } \underline{\mathbb{C}} \oplus \underline{\mathbb{C}[-2]} \text{ with Fr acting by } 1, q \text{ on these two components}$$

(recall that these are perverse sheaves so [-2] is referring to a degree shift)

$$\implies Tr(Fr) = q + 1$$